The Nuclear Shell Model

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The Nuclear Shell Model

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   - Matrix elements of transition operators
   - Effective nucleon-nucleon (NN) interactions
     (lecture of L. Bonneau on NN interaction, Thursday
      lecture of H. Molique on group theory, Friday morning)

3. No-Core Shell Model for Light Nuclei
   (lecture of R. Lazauskas on few-body problems, Tuesday)

4. Shell-model code ANTOINE (E. Caurier, F. Nowacki, IPHC Strasbourg)
   afternoon session

Lecture of A. P. Zuker “Beyond the Shell Model” on Friday afternoon
Structure of complex nuclei

- Nuclear charge (matter) density distribution $\rho_{ch}(\vec{r})$ ($\rho_m(\vec{r})$) with sharp radius $R \approx r_0A^{1/3}$.

- **Empirical evidence on the existence of an average potential and the corresponding shell structure**
  (From masses, nucleon separation energies, low-energy spectra, etc.)

- **Independent-particle motion near Fermi-level.**
  (From nucleon transfer reactions: nuclear mean-field with strong spin-orbit splitting and large shell gaps)

- **Pairing (superfluid behavior) at low excitation energy.**
  ($S_n$ versus $S_{2n}$; two-nucleon spectra, comparison of spectra of even-even and even-odd, odd-even nuclei, etc.)

- **Low-lying multipole (quadrupole) modes, vibrational or rotational energy structures.**
  (Coulomb excitation; scattering of charge particles; heavy-ion fusion-evaporation reactions, etc)

The aim of the microscopic theory is to describe these motions starting from a NN force.
The Nuclear Shell Model
Non-relativistic Hamiltonian for A nucleons

\[ \hat{H} = \sum_{i=1}^{A} \frac{\vec{p}_i^2}{2m} + \sum_{i<j=1}^{A} W(\vec{r}_i - \vec{r}_j) \]

\[ \hat{H} = \sum_{i=1}^{A} \left[ \frac{\vec{p}_i^2}{2m} + U(\vec{r}_i) \right] + \sum_{i<j=1}^{A} W(\vec{r}_i - \vec{r}_j) - \sum_{i=1}^{A} U(\vec{r}_i) , \]

Mean-field theories

Search for the most optimum mean-field potential starting from a given two-body interaction + correlations

Shell model

Schematic average potential + residual interaction
Starting point: antisymmetric product wave function

\[ \psi(1, 2, \ldots, A) = \frac{1}{\sqrt{A!}} \left| \begin{array}{cccc} \phi_{\alpha_1}(1) & \phi_{\alpha_1}(2) & \cdots & \phi_{\alpha_1}(A) \\ \phi_{\alpha_2}(1) & \phi_{\alpha_2}(2) & \cdots & \phi_{\alpha_2}(A) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{\alpha_A}(1) & \phi_{\alpha_A}(2) & \cdots & \phi_{\alpha_A}(A) \end{array} \right| \]

The best wave function is determined via a variational principle:

\[ \delta \langle \psi | H | \psi \rangle = \langle \delta \psi | H | \psi \rangle = 0 \quad \text{with} \quad \int |\phi_{\alpha_i}(\vec{r})|^2 d\vec{r} = 1. \]
Self-consistent mean-field potential

Hartree-Fock equations

\[
\left( -\frac{\hbar^2}{2m} \Delta + U_H(\vec{r}) \right) \phi_i(\vec{r}) + \int U_F(\vec{r}, \vec{r}') \phi_i(\vec{r}') d\vec{r}' = \epsilon_i \phi_i(\vec{r})
\]

Direct (Hartree) term

\[
U_H(\vec{r}) = \sum_{b \in F} \int \phi_b^*(\vec{r}') W(\vec{r}, \vec{r}') \phi_b(\vec{r}') d\vec{r}'
\]

Exchange (Fock) term

\[
U_F(\vec{r}, \vec{r}') = \sum_{b \in F} \phi_b^*(\vec{r}) W(\vec{r}, \vec{r}') \phi_b(\vec{r}')
\]

Iterative solution of Hartree-Fock equations:

\[
\begin{align*}
\phi_i^{(0)}(\vec{r}) & \quad \Downarrow \quad U_H^{(0)}(\vec{r}), U_F^{(0)}(\vec{r}, \vec{r}') \\
\phi_i^{(1)}(\vec{r}), \epsilon_i^{(1)} & \quad \Downarrow \quad U_H^{(1)}(\vec{r}), U_F^{(1)}(\vec{r}, \vec{r}') \\
\vdots & \quad \Downarrow \quad \vdots \\
\phi_i^{(\text{HF})}(\vec{r}), \epsilon_i^{(\text{HF})} & \quad \Downarrow \quad U^{(\text{HF})}(\vec{r})
\end{align*}
\]
Self-consistent mean-field potential

Direct (Hartree) term

\[ U_H(\vec{r}) = \sum_{b \in F} \int \phi^*_b(\vec{r}') W(\vec{r}, \vec{r}') \phi_b(\vec{r}') d\vec{r}' \]

\[ U_H(\vec{r}) = \int \rho(\vec{r}') W(\vec{r}, \vec{r}') d\vec{r}' \]

\[ \rho(\vec{r}) = \sum_{b \in F} |\phi_b(\vec{r})|^2 \]

Example: a delta-force

\[ W(\vec{r}, \vec{r}') \propto \delta(\vec{r} - \vec{r}') \]

\[ U_H(\vec{r}) \propto \rho(\vec{r}) \]
From the product HF wave function, a number of ground-state properties can be calculated:

\[ \Psi_{HF}(1, 2, \ldots, A) = \frac{1}{\sqrt{A!}} A \prod_{i=1}^{A} \phi^{HF}_{\alpha_i}(i) \]

\[ \langle \Psi_{HF} | \hat{H} | \Psi_{HF} \rangle = E_0 \]

\[ \langle \Psi_{HF} | \sum_{i=1}^{A} \hat{r}_i^2 | \Psi_{HF} \rangle = \langle r^2 \rangle \]

\[ \langle \Psi_{HF} | \sum_{i=1}^{A} \hat{\rho}(\vec{r}_i) | \Psi_{HF} \rangle = \rho(\vec{r}) \]

Application to open-shell nuclei and calculation excitation spectra require inclusion of correlations (pairing, etc) → beyond mean-field techniques (lecture of Ph. Quentin).
Shell model: energy matrix diagonalization

Non-relativistic Hamiltonian for A nucleons

\[ \hat{H} = \sum_{i=1}^{A} \left[ \frac{\vec{p}_i^2}{2m} + U(\vec{r}_i) \right] + \sum_{i<j=1}^{A} W(\vec{r}_i - \vec{r}_j) - \sum_{i=1}^{A} U(\vec{r}_i) \]

\( \hat{h}_i \)
\( \hat{V} \)

Construction of a basis from single-particle states

\[ \hat{h}\phi_\alpha(\vec{r}) = \varepsilon_\alpha \phi_\alpha(\vec{r}) \quad \rightarrow \quad \{ \varepsilon_\alpha, \phi_\alpha(\vec{r}) \} \]

Spherical potential \( U(\vec{r}) = U(r) \): \( \alpha = \{ n_\alpha, l_\alpha, j_\alpha, m_\alpha \} \)

\[ \phi_{nljm}(\vec{r}) = \frac{R_{nlj}(r)}{r} \left[ Y_l(\theta, \varphi) \times \chi_{1/2}^m \right](j) \]

\[ \sum_{m_l m_s} (l m_{1/2} m_s | j m) Y_{lm_l}(\theta, \varphi) \chi_{1/2}^m m_s \]
Single-particle wave functions

Radial differential equation

\[-\frac{\hbar^2}{2m} R''(r) + \frac{\hbar^2}{2m} \frac{l(l + 1)}{r^2} R(r) + [U(r)R(r) + a_{ls}f_{ls}(r)] R(r) = \varepsilon R(r)\]

Normalization condition

\[\int \left| \phi_{nlsjm}(\vec{r}) \right|^2 d\vec{r} = \sum_{m_l m_s m'_l m'_s} (l m_l \frac{1}{2} m_s jm)(l m'_l \frac{1}{2} m'_s jm) \int Y_{lm_l}(\theta, \phi) Y_{lm'_l}(\theta, \phi) d\Omega \times \langle \chi_{\frac{1}{2} m_s} | \chi_{\frac{1}{2} m'_s} \rangle \int_{0}^{\infty} |R_{nl}(r)|^2 dr = \int_{0}^{\infty} |R_{nl}(r)|^2 dr = 1\]

Parity

\[\hat{P}_{\phi_{nlsjm}(\vec{r})} = \hat{P}_{\phi_{nlsjm}(-\vec{r})} = (-1)^l \phi_{nlsjm}(\vec{r})\]
Examples of spherically-symmetric potentials

- Square-well potential + strong spin-orbit term

\[ J_{l+1/2}(kr) \]

- Harmonic oscillator potential + orbital + spin-orbit term

\[ (\nu r)^l e^{-\frac{\nu^2 r^2}{2}} L_{n-1}^{l+1/2} (\nu^2 r^2) \]

\( \nu = \sqrt{\frac{m\omega}{\hbar}} \)

N: degenerate in n, l

\[ \text{ex. } N = 4 \]

\( s_{l=0} \) (100)
\( d_{l=2} \) (93)
\( g_{l=4} \) (75.5)
Harmonic oscillator potential

\[ U(r) = \frac{m\omega^2 r^2}{2} + \alpha \vec{I} \cdot \vec{I} + \beta \vec{I} \cdot \vec{s} \]

\[ \varepsilon_N = \hbar \omega \left( 2n + l + \frac{3}{2} \right) = \hbar \omega \left( N + \frac{3}{2} \right) , \]

\[ N = 0, 1, 2, \ldots, \]
\[ l = N, N - 2, \ldots, 1 \text{ or } 0 \]
\[ n = (N - l)/2 . \]

Harmonic oscillator potential possesses many **symmetry properties** which make it a preferable choice as a basis.

\[ M. \ Mayer \ (1949) \]
\[ O. \ Haxel, \ H. \ Jensen, \ H.E. \ Suess \ (1949) \]
Isospin

The idea of W. Heisenberg: proton and neutron are considered as two states of a nucleon.

\[ \nu_{\pi \pi} \approx \nu_{\nu \nu} \approx \nu_{\pi \nu} \]

\[ \pi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \nu = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]

Isospin operators (in analogy with the Pauli matrices):

\[ \vec{t} = \frac{1}{2} \vec{\tau}, \quad \tau_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

Single-particle wave functions

\[ \phi_{\nu}(r) = \phi(\vec{r}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \phi(\vec{r}) \theta_{t=1/2,m_t=1/2}, \]

\[ \phi_{\pi}(r) = \phi(\vec{r}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \phi(\vec{r}) \theta_{t=1/2,m_t=-1/2}. \]
Isospin and classification of nuclear states

\[ \hat{T} = \sum_{i=1}^{A} \hat{t}_i, \quad \hat{T}_z = \sum_{i=1}^{A} \hat{t}_{zi}. \]

Charge independent Hamiltonian: \[ [\hat{H}, \hat{T}] = 0. \]

\[ M_T = \frac{1}{2}(N - Z), \quad \frac{1}{2}(N - Z) \leq T \leq \frac{A}{2}. \]

Realistic situation

\[ m_p \approx m_n; \quad \hat{V}_{\text{Coulomb}} = \sum_{i<j=1}^{Z} \frac{e^2}{|\vec{r}_i - \vec{r}_j|} \]

\[ \hat{V}_{\text{Coulomb}} = \hat{V}(T=0) + \hat{V}(T=1) + \hat{V}(T=2). \]

\[ E(T, M_T) = a(T) + b(T)M_T + c(T)M_T^2, \]
Two-particle wave function for identical fermions

Angular-momentum coupled state \((j_a \neq j_b)\)

\[
\Phi_{j_a(1)j_b(2);JM}(\vec{r}_1, \vec{r}_2) = \sum_{m_a m_b} (j_a m_a j_b m_b | J M) \phi_{j_a m_a}(\vec{r}_1) \phi_{j_b m_b}(\vec{r}_2) = \left[ \phi_{j_a}(\vec{r}_1) \times \phi_{j_b}(\vec{r}_2) \right]^{(J)}_M
\]

\(j_a \equiv (n_{a|a}j_a), \ J = |j_a - j_b|, |j_a - j_b| + 1, \ldots, j_a + j_b, \) while \(M = -J, -J + 1, \ldots, J - 1, J.\)

Not antisymmetric with respect to permutation of two identical fermions!

Normalized and antisymmetric state

\[
\Phi_{j_\alpha j_\beta;JM}(\vec{r}_1, \vec{r}_2) = \frac{1}{N} \sum_{m_a m_b} (j_a m_a j_b m_b | J M) \left[ \phi_{j_\alpha m_a}(\vec{r}_1) \phi_{j_\beta m_b}(\vec{r}_2) - \phi_{j_\beta m_b}(\vec{r}_1) \phi_{j_\alpha m_a}(\vec{r}_2) \right]
\]

Since the Clebsch-Gordan coefficients have the following property:

\[
(j_a m_a j_b m_b | J M) = (-1)^{j_a + j_b - J} (j_b m_b j_a m_a | J M)
\]

\[
\Phi_{j_\alpha j_\beta;JM}(\vec{r}_1, \vec{r}_2) = \frac{1}{N} \sum_{m_a m_b} [(j_a m_a j_b m_b | J M) \phi_{j_\alpha m_a}(\vec{r}_1) \phi_{j_\beta m_b}(\vec{r}_2)
- (-1)^{j_a + j_b - J} (j_b m_b j_a m_a | J M) \phi_{j_\beta m_b}(\vec{r}_1) \phi_{j_\alpha m_a}(\vec{r}_2)]
\]
Two-particle wave function for identical fermions

Normalized and antisymmetric state

\[ j_a \neq j_b \]

\[ \Phi_{j_a j_b;JM}(\vec{r}_1, \vec{r}_2) = \frac{1}{\sqrt{2}} \left\{ \left[ \phi_{j_a}(\vec{r}_1) \times \phi_{j_b}(\vec{r}_2) \right]_M^{(J)} - (-1)^{j_a+j_b+J} \left[ \phi_{j_b}(\vec{r}_1) \times \phi_{j_a}(\vec{r}_2) \right]_M^{(J)} \right\} \]

\[ j_a \equiv (n_{al}a_ja), \; J = |j_a - j_b|, |j_a - j_b| + 1, \ldots, j_a + j_b, \text{ while } M = -J, -J + 1, \ldots, J - 1, J. \]

\[ j_a = j_b = j \]

\[ \Phi_{j^2;JM}(\vec{r}_1, \vec{r}_2) = \frac{1 + (-1)^J}{2} \left[ \phi_j(\vec{r}_1) \times \phi_j(\vec{r}_2) \right]_M^{(J)} \]

Important consequence: if \( j_a = j_b = j \), then \( J = 0, 2, 4, \ldots, 2j - 1 \)

\[ (\nu 0d_{5/2})^2 : \; J = 0, 2, 4 \]
Two-particle wave function for protons and neutrons

\( j_a \neq j_b \)

\[
\Phi_{j_a j_b; JM TM_T} (\vec{r}_1, \vec{r}_2) = \left\{ \left[ \phi_{j_a} (\vec{r}_1) \times \phi_{j_b} (\vec{r}_2) \right]^M_M + (-1)^{j_a + j_b + J + T} \left[ \phi_{j_b} (\vec{r}_1) \times \phi_{j_a} (\vec{r}_2) \right]^M_M \right\} \frac{\Theta_{TM_T}}{\sqrt{2}}
\]

\[
\begin{align*}
\Theta_{1,1} &= \theta_{1/2,1/2}(1) \theta_{1/2,1/2}(2), \\
\Theta_{1,-1} &= \theta_{1/2,-1/2}(1) \theta_{1/2,-1/2}(2), \\
\Theta_{1,0} &= \left[ \theta_{1/2,1/2}(1) \theta_{1/2,-1/2}(2) + \theta_{1/2,-1/2}(1) \theta_{1/2,1/2}(2) \right] / \sqrt{2}, \\
\Theta_{0,0} &= \left[ \theta_{1/2,1/2}(1) \theta_{1/2,-1/2}(2) - \theta_{1/2,-1/2}(1) \theta_{1/2,1/2}(2) \right] / \sqrt{2},
\end{align*}
\]

\( j_a = j_b = j \)

\[
\Phi_{j^2; JM TM_T} (\vec{r}_1, \vec{r}_2) = \frac{1 - (-1)^{J + T}}{2} \left[ \phi_j (\vec{r}_1) \times \phi_j (\vec{r}_2) \right]^M_M \Theta_{TM_T}
\]

Remark: if \( j_a = j_b = j \), then \( (J + T) \) is odd!

\[
\begin{align*}
(\nu 0 d_{5/2})^2 : & \quad J = 0, 2, 4 \\
(\nu 0 d_{5/2} \pi 0 d_{5/2}) : & \quad J = 0, 2, 4 (T = 1); \quad J = 1, 3, 5 (T = 0)
\end{align*}
\]
Many-particle wave function: $J(T)$-coupled states.

J-coupled state

Consider $N$ identical fermions in a single-$j$ shell. We construct a totally antisymmetric and coupled to good $J$ $N$-nucleon wave function from a set of totally antisymmetric $(N - 1)$-nucleon wave functions coupled to all possible $J'$:

$$
\Phi^{j(N)}_{\chi J M}(\vec{r}_1, \ldots, \vec{r}_N) = \sum_{\chi' J'} \left\{ j^{N-1}(\chi' J')j \right\} j^{N} \chi J \Phi^{j(N - 1)}_{\chi' J' M'}(\vec{r}_1, \ldots, \vec{r}_{N-1}) \phi_{j m}(\vec{r}_N),
$$

where

$$
\left\{ j^{N-1}(\chi' J')j \right\} j^{N} \chi J
$$

are one-particle coefficients of fractional parentage (cfp's)

Repeat this procedure for $N'$ particles in $j'$ orbital and so on. Construct thus basis states by consecutive coupling of angular momenta and antisymmetrization.
Many-particle wave function: \( m \)-scheme basis

**Slater determinants**

\[
\Phi_\alpha(1, 2, \ldots, A) = \frac{1}{\sqrt{A!}} \begin{vmatrix}
\phi_{\alpha_1}(\vec{r}_1) & \phi_{\alpha_1}(\vec{r}_2) & \cdots & \phi_{\alpha_1}(\vec{r}_A) \\
\phi_{\alpha_2}(\vec{r}_1) & \phi_{\alpha_2}(\vec{r}_2) & \cdots & \phi_{\alpha_2}(\vec{r}_A) \\
\vdots & \vdots & \ddots & \vdots \\
\phi_{\alpha_A}(\vec{r}_1) & \phi_{\alpha_A}(\vec{r}_2) & \cdots & \phi_{\alpha_A}(\vec{r}_A)
\end{vmatrix}
\]

where \( \alpha_i = (n_i, l_i, j_i, m_i) \) and \( \alpha \) stores a set of single-particle configurations \( \{\alpha_1, \alpha_2, \ldots, \alpha_A\} \)

\[
M = \sum_{i=1}^{A} m_i
\]

Projection on \( J \) can be performed.
Consider 2 neutrons in 0\textit{f}_{7/2} orbital.

- Write down the basis in \textit{J}-coupled form.
- Write down the basis in \textit{m}-scheme.
- How many different \textit{J}-states exist in this model space?
Basis construction

- Basis in $J$-coupled scheme:

$$|(0f_{7/2})^2; J, T = 1\rangle J = 0, 2, 4, 6$$
There are 4 different $J$-states.
Solution of a many-body Schrödinger equation

Construct a basis in the valence space for each $J$

$$\Phi_k^{(J)} = \left\{ (j_a)^{n_a}(j_b)^{n_b} \right\}^{(J)}_k, \quad \hat{H}^{(0)}\Phi_k = E_k^{(0)}\Phi_k$$

Expand unknown wave function in terms of basis functions

$$\psi_p = \sum_{k=1}^{n} a_{kp}\Phi_k \quad \Rightarrow \quad \hat{H}\psi_p = E_p\psi_p \quad (\hat{H} = \hat{H}^{(0)} + \hat{V})$$

Multiplying by $\langle \Phi_l |$, we get a system of equations

$$\sum_{k=1}^{n} H_{lk} a_{kp} = E_p a_{lp}$$

$\Rightarrow$ diagonalization of the matrix

$$H_{lk} = \langle \Phi_l | \hat{H} | \Phi_k \rangle = E_k^{(0)}\delta_{lk} + V_{lk}$$

Calculate Hamiltonian matrix $H_{ij} = \langle \phi_j | H | \phi_i \rangle$

— Diagonalize to obtain eigenvalues
Basis dimension and choice of the model space

Basis dimension grows quickly

\[ \text{dim} \approx \left( \frac{\Omega_{\pi}}{N_{\pi}} \right) \left( \frac{\Omega_{\nu}}{N_{\nu}} \right) \]

Model space: a few valence orbitals beyond the closed-shell core.

Example: \(^{60}\text{Zn}\) in \(pf\)-shell

\[ \text{dim}(^{60}\text{Zn}) = \binom{20}{10} \binom{20}{10} \approx 3.4 \times 10^{10}. \]
Practical shell-model for $^{18}$O in sd-shell

$$\hat{H} = \hat{H}^{(0)} + \hat{V} = \hat{h}(1) + \hat{h}(2) + \hat{V}$$

Single-particle energies:

$$\varepsilon(0d_{5/2}) = E_B(^{17}\text{O}_9) - E_B(^{16}\text{O}_8) = -4.143 \text{ MeV}$$
$$\varepsilon(1s_{1/2}) = \varepsilon(0d_{5/2}) + E_{ex}(^{17}\text{O}; \, ^{1/2}_1^+_1) = -3.273 \text{ MeV}$$
$$\varepsilon(0d_{3/2}) = \varepsilon(0d_{5/2}) + E_{ex}(^{17}\text{O}; \, ^{3/2}_1^+_1) = 0.942 \text{ MeV}$$

Basis of states for each $(JT)$ denoted as $|j_a j_b \rangle_{JT}$:

$$|\Phi_1(0^+)\rangle \equiv |d_{5/2}^2\rangle_{01}$$
$$|\Phi_2(0^+)\rangle \equiv |s_{1/2}^2\rangle_{01}$$
$$|\Phi_3(0^+)\rangle \equiv |d_{3/2}^2\rangle_{01}$$

$$0^+, T = 1 : \Rightarrow \begin{pmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{pmatrix}$$

$$|\Phi_1(1^+)\rangle \equiv |d_{5/2} d_{3/2}\rangle_{11}$$
$$|\Phi_2(1^+)\rangle \equiv |s_{1/2} d_{3/2}\rangle_{11}$$

$$1^+, T = 1 : \Rightarrow \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}$$

and so on for $J = 2, 3, 4$. 
### Exercise 2: Energies of $0^+$ states in $^{18}$O

#### Diagonal Two-body matrix elements (TBMEs)

<table>
<thead>
<tr>
<th>Term</th>
<th>Expression</th>
<th>Energy</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_{11}$</td>
<td>$2\varepsilon(d_{5/2}) + \langle d_{5/2}^2</td>
<td>V</td>
</tr>
<tr>
<td>$H_{22}$</td>
<td>$2\varepsilon(s_{1/2}) + \langle s_{1/2}^2</td>
<td>V</td>
</tr>
<tr>
<td>$H_{33}$</td>
<td>$2\varepsilon(d_{3/2}) + \langle d_{3/2}^2</td>
<td>V</td>
</tr>
</tbody>
</table>

#### Non-diagonal TBMEs

<table>
<thead>
<tr>
<th>Term</th>
<th>Expression</th>
<th>Energy</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_{12} = H_{21}$</td>
<td>$\langle d_{5/2}^2</td>
<td>V</td>
</tr>
<tr>
<td>$H_{23} = H_{32}$</td>
<td>$\langle s_{1/2}^2</td>
<td>V</td>
</tr>
<tr>
<td>$H_{13} = H_{31}$</td>
<td>$\langle d_{5/2}^2</td>
<td>V</td>
</tr>
</tbody>
</table>
Practical shell-model for $^{18}$O in $sd$-shell

### Eigenvalues (g.s. and two excited $0^+$ states):

- $E(0^+_1) = -12.602$ MeV
- $E(0^+_2) = -8.097$ MeV
- $E(0^+_3) = 0.622$ MeV
- $E_{gs}(0^+_1) = 0$ MeV
- $E_{ex}(0^+_2) = 4.505$ MeV
- $E_{ex}(0^+_3) = 13.224$ MeV

### Eigenstates:

- $|\psi(0^+_1)\rangle = a_{11} |d_{5/2}\rangle_{01} + a_{21} |s_{1/2}\rangle_{01} + a_{31} |d_{3/2}\rangle_{01}$
- $|\psi(0^+_2)\rangle = a_{12} |d_{5/2}\rangle_{01} + a_{22} |s_{1/2}\rangle_{01} + a_{32} |d_{3/2}\rangle_{01}$
- $|\psi(0^+_3)\rangle = a_{13} |d_{5/2}\rangle_{01} + a_{23} |s_{1/2}\rangle_{01} + a_{33} |d_{3/2}\rangle_{01}$

$$\sum_k a_{kp}^2 = 1$$

### The full spectrum:

Repeat the same procedure for $J^\pi = 1^+, 2^+, 3^+, 4^+$.
Shell-model codes

\textbf{m-scheme codes}

- ANTOINE (E. Caurier)
  
  \url{http://www.iphc.cnrs.fr/nutheo/code_antoine/menu.html}
- NuShellX@MSU (W. Rae, B. A. Brown)
- MSHELL (T. Mizusaki)
- REDSTICK (W. E. Ormand, C. Johnson)
- ...

\textbf{J-coupled codes}

- NATHAN (E. Caurier, F. Nowacki)
- DUPSM (Novoselsky, Vallières)
- Ritsschil (Zwarts)
- ...

\textbf{Features}

- Matrix dimension: $\sim 10^{10}$ and beyond
- Lanczos diagonalization algorithm
- Calculation of the matrix elements on-the-fly
Lanczos algorithm

**Creation of a tri-diagonal matrix:**

\[
\begin{align*}
\hat{H}|1\rangle &= E_{11}|1\rangle + E_{12}|2\rangle \\
\hat{H}|2\rangle &= E_{21}|1\rangle + E_{22}|2\rangle + E_{23}|3\rangle \\
&\quad \ldots
\end{align*}
\]

**Matrix elements:**

\[
\begin{align*}
E_{11} &= \langle 1|\hat{H}|1\rangle \\
E_{12} &= (\hat{H} - E_{11})|1\rangle \\
E_{21} &= E_{12}, \quad E_{22} = \langle 2|\hat{H}|2\rangle \\
E_{23} &= (\hat{H} - E_{22})|2\rangle - E_{21}|1\rangle \\
&\quad \ldots
\end{align*}
\]

**How to get the lowest states converged:**

\[
\begin{pmatrix}
E_{11} & E_{12} \\
E_{21} & E_{22}
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
E_{11} & E_{12} & 0 \\
E_{21} & E_{22} & E_{23} \\
0 & E_{32} & E_{33}
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
E_{11} & E_{12} & 0 & 0 \\
E_{21} & E_{22} & E_{23} & 0 \\
0 & E_{32} & E_{33} & E_{34} \\
0 & 0 & E_{43} & E_{44}
\end{pmatrix}
\Rightarrow \ldots
\]

You may need only a few iterations to get the lowest state of a \((10^3 \times 10^3)\) matrix converged!
**Calculation of observables**

**General scheme:**

- Construct the basis: $|\Phi_k\rangle$
- Expand the wave function: $|\Psi_p\rangle = \sum_k a_{kp} |\Phi_k\rangle$ and compute the Hamiltonian matrix $\{H_{lk}\}$.
- Solution of the Shrödinger equation by Hamiltonian matrix diagonalization: $\{H_{lk}\} \Rightarrow E_p, |\Psi_p\rangle$ (coefficients $a_{kp}$)
- Calculation of matrix elements of the operators

$$T_{fi} \propto |\langle \Psi_f | \hat{O} | \Psi_i \rangle|^2$$

**Electroweak operators:**

$$\hat{O}(E, LM) = \sum_{k=1}^{A} e(k) r^L(k) Y_{LM}(\hat{r}(k))$$

$$\hat{O}(M, 1M) = \sum_{k=1}^{A} \mu_n (g_s(k) \bar{s}(k) + g_l(k) \bar{l}(k))$$

$$\hat{O}(F) = \sum_{k=1}^{A} \tau_{\pm}(k), \quad \hat{O}(GT) = \sum_{k=1}^{A} \bar{\sigma}(k) \tau_{\pm}(k) \ldots$$
Exercise 3

Electromagnetic transitions in $^{17}\text{O}$

Calculate the $B(E2; \frac{1}{2}^+_1 \rightarrow \frac{5}{2}^+_g)\text{ in }^{17}\text{O}$ modeled as a valence neutron in a $1s0d$ shell beyond the $^{16}\text{O}$ closed-shell core. We take:

$$\int_0^\infty R_{1s\frac{1}{2}}(r)r^2R_{0d\frac{5}{2}}(r)dr \approx 12\text{ fm}^2.$$ 

Compare your result to experimental value $B_{\text{exp}}(E2) = 6.3\text{ e}^2\text{.fm}^4$. What can you conclude?
Solution to exercise 3

Electromagnetic transitions in $^{17}\text{O}$

The single-particle electric multipole operator reads

$$\hat{O}(E, 2M) = e r^2 Y_{2M}(\theta, \phi),$$

The reduced probability of the $\mathcal{E}L$-transition from the initial to the final state is

$$B(\mathcal{E}L; J_i \rightarrow J_f) = \frac{1}{2J_i + 1} |\langle J_f || O(\mathcal{E}L) || J_i \rangle|^2.$$

In our case, there is one valence particle: $J_i = (1s_{1/2})$, $J_f = (0d_{5/2})$.

$$B(\mathcal{E}2; 1/2^+ \rightarrow 5/2^+) = \frac{1}{2} |\langle 0d_{5/2} || \hat{O}(\mathcal{E}2) || 1s_{1/2} \rangle|^2 =$$

$$\frac{1}{2} |\langle n_f=0, l_f=2, s_f=1/2, j_f=5/2 || e r^2 Y_2(\theta, \phi) || n_i=1, l_i=0, s_i=1/2, j_i=1/2 \rangle|^2 =$$

$$\frac{1}{2} e^2 \left( \int R_{0d_{5/2}}^*(r) r^2 R_{1s_{1/2}}(r) dr \right)^2 |\langle 2 \frac{1}{2}; 5 \frac{1}{2} || Y_2(\theta, \phi) || 0 \frac{1}{2}; 1 \frac{1}{2} \rangle|^2.$$

$$B(\mathcal{E}L; j_i \rightarrow j_f) = e^2 \frac{1}{4\pi} (2j_f + 1)(2l_i + 1)(2L + 1)(l_i0L0 || l_f0)^2 \left\{ \frac{1}{2} L \begin{array}{c} j_f \ j_i \\ l_f \ l_i \end{array} \right\}^2.$$
Electromagnetic transitions in $^{17}\text{O}$ (continued)

$$B(E2; 1/2^+ \rightarrow 5/2^+) = e^2 \frac{1}{4\pi} \langle r^2 \rangle^2 6 \times 5(0020|20)^2 \begin{pmatrix} 1/2 & 2 & 5/2 \\ 2 & 1/2 & 0 \end{pmatrix}^2 =$$

$$= 34.4 e^2 . \text{fm}^4 .$$

Experimental value $B_{\text{exp}}(E2; 1/2^+ \rightarrow 5/2^+) = 6.3 e^2 . \text{fm}^4 .$

This means that the neutron should have an effective charge: $\tilde{e}_n \approx 0.43 e.$ because we work in a severely restricted model space (one valence nucleon !).

Standard effective $E2$ and $M1$ operators

$$\tilde{e}_\pi \approx 1.5 e, \quad \tilde{e}_\nu \approx 0.5 e$$

$$\tilde{g}_s(\pi) \approx 0.7 g_s(\pi), \quad \tilde{g}_l(\pi) = g_l(\pi) = 1$$

$$\tilde{g}_s(\nu) \approx 0.7 g_s(\nu), \quad \tilde{g}_l(\nu) = g_l(\nu) = 0$$

N. A. Smirnova
The Nuclear Shell Model
Nuclear many-body problem

Non-relativistic Hamiltonian for A nucleons

\[ \hat{H} = \sum_{i=1}^{A} \frac{\hat{p}_i^2}{2m} + \sum_{i<j}^{A} \hat{W}(\vec{r}_i - \vec{r}_j) \]

\[ \hat{H} = \sum_{i=1}^{A} \left[ \frac{\hat{p}_i^2}{2m} + U(\vec{r}_i) \right] + \sum_{i<j}^{A} W(\vec{r}_i - \vec{r}_j) - \sum_{i=1}^{A} U(\vec{r}_i) \]

\[ \hat{H} = \hat{H}^{(0)} + \hat{V} \]

The residual interaction is assumed to have a two-body form

\[ \hat{V} = \sum_{i<j=1}^{A} \hat{V}(\vec{r}_i - \vec{r}_j) \]

One and two-body Hamiltonian

\[ \hat{H} = \sum_{i=1}^{A} \hat{h}(\vec{r}_i) + \sum_{i<j=1}^{A} \hat{V}(\vec{r}_i - \vec{r}_j) \]
Creation and annihilation operators

\[ |\alpha\rangle = a_\alpha^\dagger |0\rangle \quad \langle \alpha | = \langle 0 | a_\alpha \]

Wave function of a fermion in a quantum state \( \alpha \) in coordinate space:

\[ \langle \vec{r} | \alpha \rangle = \phi_\alpha (\vec{r}) \]

(Anti-)commutation relations:

\[
\begin{align*}
\{ a_\alpha^\dagger, a_\beta \} &= a_\alpha^\dagger a_\beta + a_\beta a_\alpha^\dagger = \delta_{\alpha\beta} \\
\{ a_\alpha^\dagger, a_\beta^\dagger \} &= \{ a_\alpha, a_\beta \} = 0
\end{align*}
\]

Normalized and antisymmetric \( A \)-fermion state:

\[ |\alpha_1 \alpha_2 \ldots \alpha_A \rangle = a_{\alpha_A}^\dagger a_{\alpha_{A-1}}^\dagger \ldots a_{\alpha_2}^\dagger a_{\alpha_1}^\dagger |0\rangle \]
Operators in the occupation-number formalism

One-body operators

\[ \hat{O} = \sum_{k=1}^{A} \hat{O}(\vec{r}_k) \]

\[ \langle \alpha | \hat{O} | \beta \rangle = \int \phi_{\alpha}^*(\vec{r}) \hat{O}(\vec{r}) \phi_{\beta}(\vec{r}) d\vec{r} \]

Second-quantized form of the one-body operator \( \hat{O} \):

\[ \hat{O} = \sum_{\alpha \beta} \langle \alpha | \hat{O} | \beta \rangle a_{\alpha}^\dagger a_{\beta} \]

For example, the number operator reads

\[ \hat{N} = \sum_{\alpha \beta} \langle \alpha | \hat{1} | \beta \rangle a_{\alpha}^\dagger a_{\beta} = \sum_{\alpha} a_{\alpha}^\dagger a_{\alpha} \]
Operators in the occupation-number formalism

Symmetric two-body operator acting on an $A$-fermion system

\[ \hat{T} = \sum_{j<k=1}^{A} \hat{T}(\vec{r}_k, \vec{r}_j) \]

\[
\langle \alpha\beta | \hat{T} | \gamma\delta \rangle = \int \phi^*_\alpha(\vec{r}_1)\phi^*_\beta(\vec{r}_2) \hat{T}(\vec{r}_1, \vec{r}_2) \left( 1 - \hat{P}_{12} \right) \phi_\gamma(\vec{r}_1)\phi_\delta(\vec{r}_2) d\vec{r}_1 d\vec{r}_2 ,
\]

Second-quantized form of the two-body operator $\hat{T}$:

\[ \hat{T} = \frac{1}{4} \sum_{\alpha\beta\gamma\delta} \langle \alpha\beta | \hat{T} | \gamma\delta \rangle a^\dagger_\alpha a^\dagger_\beta a_\delta a_\gamma \]
Nuclear many-body Hamiltonian in the occupation-number formalism

Non-relativistic Hamiltonian for $A$ nucleons

$$\hat{H} = \sum_{i=1}^{A} \hat{h}(\vec{r}_i) + \sum_{i<j=1}^{A} \hat{V}(\vec{r}_i - \vec{r}_j) = \hat{H}^{(0)} + \hat{V}$$

$$\downarrow$$

$$\hat{H} = \sum_{\alpha} \varepsilon_{\alpha} a^+_\alpha a_\alpha + \frac{1}{4} \sum_{\alpha \beta \gamma \delta} \langle \alpha \beta | V | \gamma \delta \rangle a^+_\alpha a^+_\beta a_\delta a_\gamma ,$$

\begin{align*}
\text{one-body term} & \quad \text{two-body term}
\end{align*}

Two-body term in a $JT$-coupled form

$$\hat{V} = -\frac{1}{4} \sum_{j_\alpha j_\beta \gamma j_\delta} \langle j_\alpha j_\beta | V | j_\gamma j_\delta \rangle_{JT} \sqrt{(1 + \delta_{\alpha \beta})(1 + \delta_{\gamma \delta})}$$

$$\left[ \hat{a}^+_j a^+_i \right]^{(JT)} \left[ \tilde{a}_j \tilde{a}_i \right]^{(JT)}^{(00)}$$
Necessary ingredients

Single-particle energies

\[ \varepsilon_\alpha \]

from experimental spectra of \( A_{\text{core}} \) plus a neutron or a proton

Two-body matrix elements (TBMEs)

\[ \langle j_\alpha j_\beta | V | j_\gamma j_\delta \rangle_{JT} \]

from theory?
Bare nucleon-nucleon (NN) interaction

The NN interaction between two nucleons in the vacuum: NN scattering data, deuteron bound states properties.

Elastic scattering in momentum space (Yukawa)

\[ V_{\pi NN}(1, 2) = \frac{g^2_{\pi NN}}{4M^2} \frac{(\vec{\sigma}_1 \cdot \vec{q}) (\vec{\sigma}_2 \cdot \vec{q})}{\vec{q}^2 + m^2_\pi} \]

Potential (Fourier transform) in coordinate space

\[ V_{\pi NN}^{OPEP}(1, 2) = \frac{g^2_{\pi NN}}{4M^2} \frac{m^3_\pi}{12} \left\{ \vec{\sigma}_1 \cdot \vec{\sigma}_2 \right\} 
\quad + \left( 1 + \frac{3}{m_\pi r} + \frac{3}{(m_\pi r)^2} \right) \left( 3\vec{\sigma}_1 \cdot \vec{r} \vec{\sigma}_2 \cdot \vec{r} - \vec{\sigma}_1 \cdot \vec{\sigma}_2 \right) \right\} \frac{e^{-m_\pi r}}{m_\pi r} \]

Meson-exchange theories of NN potential: high-precision potentials (CD-Bonn, AV18, etc)
Concept of effective interaction (operators)

Effective in-medium nucleon-nucleon interaction

- In-medium effects (renormalization of the hard core)
- Truncated model space

\[ \hat{H}\Psi = (\hat{H}^{(0)} + \hat{V})\Psi = E\Psi, \]

True wave function:

\[ \Psi = \sum_{k=1}^{\infty} a_k \Phi_k. \]

Wave function in a model space:

\[ \Psi' = \sum_{k=1}^{M} a_k \Phi_k. \]

\[ \langle \Psi' | \hat{H}_{\text{eff}} | \Psi' \rangle = \langle \Psi | \hat{H} | \Psi \rangle = E \]

\[ \langle \Psi' | \hat{O}_{\text{eff}} | \Psi' \rangle = \langle \Psi | \hat{O} | \Psi \rangle \]

Approaches to the problem: **phenomenological** or **microscopic**.
Effective Interaction

Practical approaches to effective interaction

- **Schematic** interaction (parametrized interaction between two nucleons in a nuclear medium)

- **Phenomenological** interaction (Fit of TBME’s to energy levels of nuclei to be described within the chosen model space)

- **Microscopic** interaction (derived from a bare NN-force)
A few parameters (interaction strengths) are fitted to reproduce energy levels in a certain region of (a few) neighboring nuclei ⇒ local description only!
Exercise 4: TBMEs of the $\delta$-force

Multipole expansion of the delta-function

$$V(1, 2) = -V_0 \delta(\vec{r}_1 - \vec{r}_2),$$

$$\delta(\vec{r}_1 - \vec{r}_2) = \sum_k \frac{\delta(r_1 - r_2)}{r_1 r_2} \frac{2k + 1}{4\pi} P_k(\cos \theta_{12})$$

Diagonal TBMEs between normalized and antisymmetric states

$$\langle j_1 j_2 | V | j_1 j_2 \rangle_{JT} = I (2j_1 + 1)(2j_2 + 1) \left( \begin{array}{ccc} j_1 & j_2 & J \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{array} \right)^2 \frac{1 + (-1)^{l_1 + l_2 + J}}{2}$$

$$\langle j^2 | V | j^2 \rangle_{JT} = I (2j + 1)^2 \left( \begin{array}{ccc} j & j & J \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{array} \right)^2 , \quad \text{if } j_1 = j_2 = j$$

$$I = \frac{1}{4\pi} \int_0^\infty \frac{1}{r^2} [R_{n_1 l_1}(r) R_{n_2 l_2}(r)]^2 \, dr$$
Example 1: $^{210}$Pb in $(\nu 0 h_{9/2})^2$

$$V_{\delta}(1,2) = -V_0 \delta(\vec{r}_1 - \vec{r}_2)$$

$$V_{\text{pairing}}(1,2) = -G \hat{S}_+ \cdot \hat{S}_-$$

$$\langle j_a^2 | V_{\text{pairing}}(1,2) | j_b^2 \rangle_{01} = -(-1)^{l_a+l_b} \frac{1}{2} G \sqrt{(2j_a+1)(2j_b+1)}$$

![Graphical representation of energy levels and transitions in $^{210}$Pb]
Example 2: $^{20}\text{Ne}$ and SU(3) model of Elliott

\[ \hat{H} = \sum_{i=1}^{A} \left[ -\frac{p_i^2}{2m} + \frac{1}{2} m\omega^2 r_i^2 \right] - \chi Q \cdot Q \]

$Q$ is an algebraic quadrupole operator ($Q_\mu, L_\mu$ are SU(3) generators) J.P.Elliott (1958)

Group-theoretical classification of nuclear states (analytical solution) — see lecture of H. Molique.

Rotational classification of nuclear states as mixing of many spherical configurations

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Empirical interaction (least-squares-fit method)

All TBME's $\langle j_{ab} | V | j_{cd} \rangle_{JT}$ are considered as free parameters!

Examples:
- $0p$-shell: $^4\text{He} - ^{16}\text{O}$ (15 TBME's) Cohen, Kurath (1965)
- $1s0d$-shell: $^{16}\text{O} - ^{40}\text{Ca}$ (63 TBME's) Brown, Wildenthal (1988)
- $1p0f$-shell: $^{40}\text{Ca} - ^{80}\text{Zr}$ (195 TBME's) Honma et al (2002, 2004)
A bare NN potential (CD-Bonn, AV18, chiral N3LO, etc) requires regularization and modification to be applied for many-body calculations in a restricted model space. 

Renormalization schemes (*see lecture of L. Bonneau*)

- $G$ matrix followed by the many-body perturbation theory
- $V_{\text{low}-k}$
- SRG (IM-SRG)
- Okubo-Lee-Suzuki transformation

Successful, but still lack precision of the empirical interactions, mainly due to behavior of centroids. One of possible reasons: absence of 3N forces (*A. Poves, A.P. Zuker, 1981; A.P. Zuker, 2003*)
Modern nuclear shell model and beyond

Interacting shell model

Oscillator-based shell model with accurate realistic interactions formulated in one or two harmonic-oscillator shells model spaces (large-scale diagonalization).

- Detailed information on individual states and transitions at low energies
- Conservation of principal symmetries

Numerous applications to nuclear structure, weak interaction and astrophysics

E. Caurier et al, Rev. Mod. Phys. 77, 427 (2005)
State-of-the-art calculations: backbending in $^{48}\text{Cr}$

For $J<10$:

$$E_J \sim J(J + 1)$$

$$Q_0 = \frac{(J + 1)(2J + 3)}{3K^2 - J(J + 1)} Q_{\text{spec}}(J), \ K \neq 1$$

$$B(E2; J \rightarrow J - 2) = \frac{5}{16\pi} e^2 |(JK20 | J - 2, K)^2 Q_0^2$$

$J<10$: collective rotation

$J=10-12$: backbending phenomenon (competition between rotation and alignment of $0f_{7/2}$ particles)

$J>12$: spherical states

KB3 (semi-empirical interaction in pf-shell model space)

Strasbourg-Madrid

E. Caurier et al, Rev. Mod. Phys. 77 (2005) 427
Intruder np-nh configurations can lead even to superdeformation!

$[\text{sd}]^{16}[\text{pf}]^0 - 0\text{p}0\text{h} -$ spherical configuration

$[\text{sd}]^{12}[\text{pf}]^4 - 4\text{p}4\text{h} -$ deformed configuration

State-of-the-art calculations: mirror bands in $A = 51$

S. Lenzi, A. Zuker, E. Caurier et al
Non-core shell model

Non-relativistic Hamiltonian for $A$ nucleons in many $N\hbar\Omega$ harmonic oscillator space

$$\hat{H} = \sum_{i=1}^{A} \frac{\vec{p}_i^2}{2m} + \sum_{i<j=1}^{A} W(\vec{r}_i - \vec{r}_j)$$

Problem: excitation of the center-of-mass of the system.

Center-of-mass coordinates

$$\vec{R} = \frac{1}{A} \sum_{i=1}^{A} \vec{r}_i ; \quad \vec{P} = \sum_{i=1}^{A} \vec{p}_i$$

Translational-invariant Hamiltonian

$$\hat{H} = \sum_{i=1}^{A} \frac{\vec{p}_i^2}{2m} - \frac{\vec{P}^2}{2mA} + \sum_{i<j=1}^{A} W(\vec{r}_i - \vec{r}_j)$$
Separation of the harmonic oscillator Hamiltonian into center-of-mass and intrinsic Hamiltonians

\[ H_{ho} = \sum_{i=1}^{A} \left[ \frac{\vec{p}_i^2}{2m} + \frac{m\Omega^2 \vec{r}_i^2}{2} \right] \]

\[ = \frac{1}{2mA} \sum_{i<j=1}^{A} (\vec{p}_i - \vec{p}_j)^2 + \frac{m\Omega^2}{2A} \sum_{i<j=1}^{A} (\vec{r}_i - \vec{r}_j)^2 + \left( \frac{\vec{P}^2}{2mA} + \frac{mA\Omega^2 \vec{R}^2}{2} \right) \]

\[ \hat{H}_{int} \]

\[ \hat{H}_{CoM} \]
No-Core Shell Model

Translational-invariant Hamiltonian

\[ \hat{H} = \sum_{i=1}^{A} \left[ \frac{\vec{p}_i^2}{2m} + \frac{m\Omega^2 \vec{r}_i^2}{2} \right] - \frac{\vec{P}^2}{2mA} + \sum_{i<j=1}^{A} W(\vec{r}_i - \vec{r}_j) - \sum_{i=1}^{A} \frac{m\Omega^2 \vec{r}_i^2}{2} \]

\[ \hat{H} = \sum_{i=1}^{A} \left[ \frac{\vec{p}_i^2}{2m} + \frac{m\Omega^2 \vec{r}_i^2}{2} \right] + \sum_{i<j=1}^{A} W(\vec{r}_i - \vec{r}_j) - \frac{m\Omega^2}{2A} \sum_{i<j=1}^{A} (\vec{r}_i - \vec{r}_j)^2 - \hat{H}_{\text{CoM}} \]

Separation of the Center-of-Mass motion

\[ \hat{H}^\Omega = \hat{H} + \beta \left( \hat{H}_{\text{CoM}} - \frac{3}{2} \hbar\Omega \right) \]

See lecture of R. Lazauskas for alternative methods.
The Nuclear Shell Model

- The Nuclear Shell Model is a powerful microscopic approach to nuclear structure
- Very good description of energies and transitions at low energies
- High predictive power.
- Important applications:
  - structure of nuclei far from stability (proton rich or neutron-rich nuclei)
  - calculation of weak interaction processes on nuclei for the tests of the Standard Model
  - calculation of the nuclear structure input (masses, half-lives, reaction rates, etc) relevant for astrophysical simulations (r-processes, rp-process, etc..)
Some references on the Shell model theory

Books


Reviews