Theoretical models for nuclear astrophysics

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1. Introduction
1. Introduction

Goal of nuclear astrophysics: understand the abundances of the elements

- H, \(^4\)He most abundant (~75%, ~25%)
- « Gap » between \(A=4\) and \(A=12\): no stable element with \(A=5\) and \(8\)
- Iron peak (very stable)

Nuclear astrophysics

- Masses
- Cross sections
- \(\beta\) lifetimes
- Fission barriers
- Etc…

masses

Stellar model

astrophysicist

big bang

fusi
1. Introduction

- Years ~ 1940-50: Hoyle, Gamow
  Role of nuclear reactions in stars
  - Energy production
  - Nucleosynthesis (Hoyle state in $^{12}$C)

- 1957: B$_2$FH: Burbidge, Burbidge, Fowler, Hoyle (Rev. Mod. Phys. 29 (1957) 547)
  Cycles: pp chain: converts $4p \rightarrow 4\text{He}$
  CNO cycle: converts $4p \rightarrow 4\text{He}$ (via $^{12}$C)
  s (slow) process: $(n,\gamma)$ capture followed by $\beta$ decay
  r (rapid) process: several $(n,\gamma)$ captures
  p (proton) process: $(p,\gamma)$ capture

- Nucleosynthesis:
  Primordial (Bigbang): 3 first minutes of the Universe
  Stellar: star evolution, energy production

- Essentially two (experimental) problems in nuclear astrophysics
  Low energies $\rightarrow$ very low cross sections (Coulomb barrier)
  Need for radioactive beams
  $\Rightarrow$ in most cases a theoretical support is necessary (data extrapolation)
Reaction networks: set of equations with abundances of nucleus m: $Y_m$

\[
\frac{dY_m}{dt} = -\lambda_m Y_m + \sum_k \lambda_m^{(m)} Y_k - \sum_k Y_m Y_k [m/k]^{(m+k)} + \sum_{k,l} Y_k Y_l [kl]^{(m)}
\]

→ Destruction of m by β decay: $\lambda_m = 1/\tau_m$

→ Production of m by β decay from elements k

→ Destruction of m by reaction with k

→ Production of m by reaction k+l→m

with $[kl]^{(m)} \sim <\sigma v>$, $<\sigma v>$=reaction rate (strongly depends on temperature)

In practice:

• Many reactions are involved (no systematics)

• $\sigma$ must be known at very low energies $\rightarrow$ very low cross sections

• Reactions with radioactive elements are needed

• At high temperatures: high level densities $\rightarrow$ properties of many resonances needed
Specificities of nuclear astrophysics

- low energies (far below the Coulomb barrier)
  - small cross sections
  - (in general not accessible in laboratories at stellar energies)
  - low angular momenta (selection of resonances)

- radioactive nuclei
  - need for radioactive beams (\(^{13}\text{N},\,^{18}\text{F},\,^{7}\text{Be},\,\ldots\))

- different types of reactions:
  - transfer \((\alpha,\text{n}),\,(\alpha,\text{p}),\,(\text{p},\alpha),\,\text{etc}...\)
  - radiative-capture: \((\text{p},\gamma),\,(\alpha,\gamma),\,(\text{n},\gamma),\,\text{etc}...\)
  - Weak processes: \(\text{p}+\text{p} \rightarrow \text{d} + \text{e}^+ + \nu\)
  - fusion: \(^{12}\text{C}+^{12}\text{C},\,^{16}\text{O}+^{16}\text{O},\,\text{etc}...\)

- different situations
  - capture, transfer
  - resonant, non resonant
  - low level density, high level density
  - peripheral, internal processes

- different approaches, for theory and for experiment
1. Introduction

Some key reactions

• Triple $\alpha$, $^{12}\text{C}(\alpha,\gamma)^{16}\text{O}$: He burning
• $^{7}\text{Be}(p,\gamma)^{8}\text{B}$: solar neutrino problems
• $^{18}\text{F}(p,\alpha)^{15}\text{O}$: nova nucleosynthesis
• Etc...
2. Low-energy cross sections

- Definitions
- General properties
- S-factor
### Types of reactions: general definitions valid for all models

<table>
<thead>
<tr>
<th>Type</th>
<th>Example</th>
<th>Origin</th>
</tr>
</thead>
<tbody>
<tr>
<td>Transfer</td>
<td>$^{3}\text{He}(^{3}\text{He},2p)^{\alpha}$</td>
<td>Strong</td>
</tr>
<tr>
<td>Radiative capture</td>
<td>$^{2}\text{H}(p,\gamma)^{3}\text{He}$</td>
<td>Electromagnetic</td>
</tr>
<tr>
<td>Weak capture</td>
<td>$p+p \rightarrow d + e^{+} + \nu$</td>
<td>Weak</td>
</tr>
</tbody>
</table>

*cross section decreases*
2. Low-energy cross sections

- **Transfer**: \( A+B \rightarrow C+D \) (\( \sigma_t \), strong interaction, example: \(^3\)He(d,p\(^4\)He)

\[
\sigma_{t,c\rightarrow c'}(E) = \frac{\pi}{k^2} \sum_{J,\pi} \frac{2J+1}{(2I_1+1)(2I_2+1)} |U_{cc'}^{J\pi}(E)|^2
\]

\( U_{cc'}^{J\pi}(E) \) = collision (scattering) matrix (obtained from scattering theory \( \rightarrow \) various models)

\( c, c' \) = entrance and exit channels

Transfer reaction:
Nucleons are transfered

Scattering energy \( E \)

A+B threshold, ex: \(^3\)He+d

C+D threshold, ex: \(^4\)He+p

Compound nucleus, ex: \(^5\)Li
2. Low-energy cross sections

- Radiative capture: $A+B \rightarrow C+\gamma$ ($\sigma_C$, electromagnetic interaction, example: $^{12}\text{C}(p,\gamma)^{13}\text{N}$)

\[ \sigma_C^{J_f\pi_f}(E) \sim \sum_{\lambda} \sum_{J_i\pi_i} k_{\gamma}^{2\lambda+1} |\langle \Psi_{J_f\pi_f} | M_{\lambda} | \Psi_{J_i\pi_i}(E) \rangle|^2 \]

$J_f\pi_f$=final state of the compound nucleus C
$\Psi_{J_i\pi_i}(E)$=initial scattering state of the system (A+B)
$M_{\lambda\mu}$=electromagnetic operator (electric or magnetic): $M_{\lambda\mu} \sim e r^{\lambda} Y_{\lambda}^{\mu}(\Omega_{r})$

Capture reaction:
A photon is emitted

Long wavelength approximation:
Wave number $k_{\gamma} = E_{\gamma}/\hbar c$, wavelength: $\lambda_{\gamma} = 2\pi/k_{\gamma}$
Typical value: $E_{\gamma} = 1$ MeV, $\lambda_{\gamma} \approx 1200$ fm $>>$ typical dimensions of the system ($R$)
$\rightarrow k_{\gamma} R \ll 1$ = Long wavelength approximation
2. Low-energy cross sections

initial state $E>0$, contains all $J_i\pi_i$,

A+B threshold, ex: $^{12}$C+p

final states $E_f<0$, specific $J_f\pi_f$

\[
\sigma_C^{J_f\pi_f}(E) \sim \sum_{J_i\pi_i} \sum_{\lambda} k_\gamma^{2\lambda+1} |<\psi_{J_f\pi_f}|\mathcal{M}_\lambda|\psi_{J_i\pi_i}(E)>|^2
\]

- $k_\gamma = (E - E_f)/\hbar c = \text{photon wave number}$
- In practice
  - Summation over $\lambda$ limited to 1 term (often E1, or E2/M1 if E1 is forbidden)
    \[
    \frac{E_2}{E_1} \sim (k_\gamma R)^2 \ll 1 \text{ (from the long wavelength approximation)}
    \]
  - Summation over $J_i\pi_i$ limited by selection rules
    \[
    |J_i - J_f| \leq \lambda \leq J_i + J_f
    \]
    \[
    \pi_i\pi_f = (-1)^\lambda \text{ for electric, } \pi_i\pi_f = (-1)^{\lambda+1} \text{ for magnetic}
    \]
2. Low-energy cross sections

- **Weak capture** \((p+p \rightarrow d+\nu+e^-)\): tiny cross section \(\rightarrow\) no measurement (only calc.)

\[
\sigma^{J_f\pi_f}_W (E) \sim \sum_{J_i\pi_i} \left| \langle \Psi^{J_f\pi_f} | O_{\beta} | \Psi^{J_i\pi_i} (E) \rangle \right|^2
\]

  - Calculations similar to radiative capture
  - \(O_{\beta} = \text{Fermi} (\sum_{i} t_{i\pm}) \text{ and Gamow-Teller} (\sum_{i} t_{i\pm} \sigma_{i}) \) operators

- **Fusion**: similar to transfer, but with many output channels
  - \(\rightarrow\) statistical treatment
  - \(\rightarrow\) optical potentials
2. Low-energy cross sections

General properties

- Scattering energy $E$: wave function $\Psi_i(E)$ common to all processes
- Reaction threshold

Cross sections dominated by Coulomb effects
Sommerfeld parameter $\eta = \frac{Z_1 Z_2 e^2}{\hbar v}$

Coulomb functions at low energies

- $F_\ell(\eta, x) \rightarrow \exp(-\pi \eta) F_\ell(x)$,
- $G_\ell(\eta, x) \rightarrow \exp(\pi \eta) G_\ell(x)$,

Coulomb effect: strong $E$ dependence: $\exp(2\pi \eta)$

- Neutrons: $\sigma(E) \sim 1/v$

Strong $\ell$ dependence
Centrifugal term: $\sim \frac{\hbar^2 \ell(\ell+1)}{2\mu r^2}$

→ stronger for nucleons ($\mu \approx 1$) than for $\alpha$ ($\mu \approx 4$)
2. Low-energy cross sections

General properties: specificities of the entrance channel \(\rightarrow\) common to all reactions

- All cross sections (capture, transfer) involve a summation over \(\ell\):
  \[
  \sigma(E) = \sum_\ell \sigma_\ell(E)
  \]
- The partial cross sections \(\sigma_\ell(E)\) are proportional to the penetration factor:
  \[
  P_\ell(E) = \frac{ka}{F_\ell(ka)^2 + G_\ell(ka)^2}
  \]
  \((a = \text{typical radius})\)

**Consequences**
- \(\ell > 0\) are often negligible at low energies
- \(\ell = \ell_{\text{min}}\) is dominant (often \(\ell_{\text{min}} = 0\))
- For \(\ell = 0\), \(P_0(E) \sim \exp(-2\pi\eta)\)

**Astrophysical S factor:**
\[
S(E) = \sigma(E)E\exp(2\pi\eta)
\]
(Units: \(E^2\)L: MeV-barn)
- removes the coulomb dependence \(\rightarrow\) only nuclear effects
- weakly depends on energy \(\rightarrow\)
  \[
  \sigma(E) \approx S_0 \exp(-2\pi\eta)/E
  \]
  (any reaction at low \(E\))
non resonant: \( S(E) = \sigma(E)E \exp(2\pi\eta) \)

Example: \(^{3}\text{He}(\alpha,\gamma)^{7}\text{Be}\) reaction

- Cross section \( \sigma(E) \) Strongly depends on energy
- Logarithmic scale

S factor
- Coulomb effects removed
- Weak energy dependence
- Linear scale
2. Low-energy cross sections

**Resonant cross sections**: Breit-Wigner form

\[
\sigma_R(E) \approx \frac{\pi}{k^2} \frac{2J_R + 1}{(2I_1 + 1)(2I_2 + 1)} \frac{\Gamma_1(E)\Gamma_2(E)}{(E_R - E)^2 + \Gamma^2/4}
\]

- \(J_R, E_R\) = spin, energy of the resonance
- Valid for any process (capture, transfer)
- Valid for a single resonance \(\rightarrow\) several resonances need to be added (if necessary)

\(\Gamma_1\) = Partial width in the entrance channel (strongly depends on \(E, \ell\))

\[
\Gamma_1(E) = 2\gamma_1^2 P_\ell(E)
\]

with \(\gamma_1^2\) = reduced width (does not depend on \(E\))

\[
P_\ell(E) \sim \exp(-2\pi\eta)
\]

A resonance at low energies is always narrow (role of \(P_\ell(E)\))

\(\Gamma_2\) = Partial width in the exit channel (weakly depends on \(E, \ell\))

- Transfer: \(\Gamma_2(E) = 2\gamma_2^2 P_\ell_f(E + Q)\) (in general \(Q \gg E \rightarrow P_\ell_f(E + Q)\) almost constant)
- Capture: \(\Gamma_2(E) \sim (E - E_f)^{2\lambda + 1} B(E\lambda) \rightarrow\) weak energy dependence

- \(S\) factor near a resonance \(S(E) = \sigma(E)E\exp(2\pi\eta)\)

\[
S_R(E) \sim \frac{\gamma_1^2\Gamma_2}{(E_R - E)^2 + \Gamma^2/4} P_\ell(E) \exp(2\pi\eta)
\]

\(\rightarrow\) Simple estimate at low \(E\) (at the Breit-Wigner approximation)
2. Low-energy cross sections

\[ S_R(E) \sim \frac{\gamma_1^2 \Gamma_2}{(E_R - E)^2 + \Gamma^2 / 4} P_\ell(E) \exp(2\pi \eta) \]
\[ \sim \frac{\gamma_2^2 \Gamma_2}{(E_R - E)^2 + \Gamma^2 / 4} \]

- For \( \ell = 0 \): \( P_0(E) \exp(2\pi \eta) \sim \text{constant} \)
- For \( \ell > 0 \), \( P_\ell(E) \ll P_0(E) \)
  \( \Rightarrow \ell > 0 \) resonances are suppressed

In \(^{12}\text{C}(p,\gamma)^{13}\text{N}:\)
- Resonance 1/2\(^+\): \( \ell = 0 \)
- Resonances 3/2\(^-\), 5/2\(^+\) \( \ell = 1,2 \) \( \Rightarrow \) negligible

Note: BW is an approximation
- Neglects background, external capture
- Assumes an isolated resonance
- Is more accurate near the resonance energy
2. Low-energy cross sections

$^3\text{He}(d,p)^4\text{He}$: isolated resonance in a transfer reaction

$^3\text{He}+d$ $\rightarrow$ $^3/2^+\rightarrow^3/2^+$ $Q=18.4$ MeV

$\alpha+p$ $\rightarrow$ $1/2^-\rightarrow 3/2^-$

3/2+ resonance:

- Entrance channel: spin $S=1/2, 3/2$, parity $+$ $\rightarrow \ell = 0, 2$
- Exit channel: spin $S=1/2$, parity $+$ $\rightarrow \ell = 1$
2. Low-energy cross sections

Breit Wigner approximation

\[
\sigma_{dp}(E) \approx \frac{\pi}{k^2} \frac{(2J_R + 1)}{(2I_1 + 1)(2I_2 + 1)} \frac{\Gamma_d(E)\Gamma_p(E)}{(E_R - E)^2 + \Gamma^2/4}
\]

Width at half maximum = total width \(\Gamma\)

Amplitude: \(\sim \Gamma_d \Gamma_p / \Gamma^2\)

S(E) constant for \(\ell = 0\)
2. Low-energy cross sections

Two comments:
1. Selection of the main resonances
2. Going beyond the Breit-Wigner approximation

1. Selection of the main resonances

\[ {}^{11}\text{C}(p,\gamma){}^{12}\text{N} \quad \text{(spin } {}^{11}\text{C}=3/2^-) \]
- Resonance 2\(^-\): \(\ell = 0, \text{ E1} \)
- Resonance 2\(^+\): \(\ell = 1, \text{ E2/M1} \)
\(\rightarrow\) negligible

\[ {}^{18}\text{F}(p,\alpha){}^{15}\text{O} \quad \text{(spin } {}^{18}\text{F}=1^-) \]
- Many resonances
- Only \(\ell = 0\) resonances are important
\(\rightarrow J = 1/2^+, 3/2^+\) only

\(\rightarrow\) In general a small number of resonances play a role
2. Low-energy cross sections

2. Going beyond the Breit-Wigner approximation

- How to go beyond the BW approximation?
- Problem of vocabulary
  - Direct capture
  - External capture
  - Non-resonant capture = « direct » capture
  → confusion!

- External capture \( \sigma(E) = |M_{int} + M_{ext}|^2 \)
  With \( \sigma_{BW}(E) = |M_{int}|^2 \)
  \( M_{ext} \sim C \), with C=Asymptotic Normalization Constant (ANC) is needed

- Non resonant capture: \( \sigma(E) = \sum_{\ell} \sigma_{\ell}(E) = \sigma_{R}(E) + \sum_{\ell \neq R} \sigma_{\ell}(E) \)
  → scanning the resonance is necessary
2. Low-energy cross sections

Many different situations

- **Transfer cross sections** (*strong interaction*)
  - Non resonant:
  - Resonant, with \( l_R = l_{\text{min}} \):
  - Resonant, with \( l_R > l_{\text{min}} \):
  - Multiresonance:

- **Capture cross sections** (*electromagnetic interaction*)
  - Non resonant:
  - Resonant, with \( l_R = l_{\text{min}} \):
  - Resonant, with \( l_R > l_{\text{min}} \):
  - Multiresonance:
  - Subthreshold state:

- **Weak capture cross sections** (*weak interaction*)
  - Non resonant
\[ ^6\text{Li}(p,\gamma)^7\text{Be} \]

\[ ^7\text{Be}(p,\gamma)^8\text{B} \]

\[ ^{12}\text{C}(p,\gamma)^{13}\text{N} \]

Energy level schemes:
- Non resonant
- Resonant \( \ell_{\text{min}}=0 \), \( E1 \)
- Resonant \( \ell_{\text{R}}=1 \), \( M1 \)
- Resonant \( \ell_{\text{min}}=0 \)
- \( \ell_{\text{R}}=0 \)
2. Low-energy cross sections

For the reaction $^{12}\text{C}(\alpha,\gamma)^{16}\text{O}$, the S-factor (keV-b) is shown as a function of energy (MeV). The S-factor peaks at different energies, indicating resonances at specific energies:

- States at $2^+$ and $1^-$.

For the reaction $^{22}\text{Ne}(\alpha,n)^{25}\text{Mg}$, the S-factor (MeV-b) is also shown as a function of energy (MeV). This reaction involves a multiresonant mechanism:

- 20 states
- 125 states

The general situation for heavy nuclei involves a multiresonant mechanism, where the S-factor shows a complex behavior due to the presence of many states.

Subthreshold states: $2^+$, $1^-$
3. Reaction rates

1. Definitions
2. Gamow peak
3. Non-resonant rates
4. Resonant rates
3. Reaction rates

1. Definition

Quantity used in astrophysics: reaction rate (integral over the energy E)

\[ N_A < \sigma v > = N_A \int \sigma(E) \nu N(E,T) dE \]

- Definition valid for resonant and non–resonant reactions
- \( N_A \) = Avogadro number
- \( T = \) temperature, \( \nu = \) velocity, \( k_B = \) Boltzmann constant \( (k_B \sim \frac{1}{11.6} \text{MeV}/10^9 K) \)
- \( N(E,T) = \left( \frac{8E}{\pi \mu m_N (k_B T)^3} \right)^{1/2} \exp \left( -\frac{E}{k_B T} \right) = \) Maxwell–Boltzmann distribution
- \( \frac{1}{N_A < \sigma v >} = \) typical reaction time
- 2 approaches
  - numerical
  - analytical: non–resonant and resonant reactions treated separately

⇒ essentially two energy dependences: \( \exp \left( -\frac{E}{k_B T} \right) \): decreases with E
  \( \exp(-2\pi \eta) \): increases with E
2. The Gamow peak

Defines the energy range relevant for the reaction rate (non-resonant reactions)

Gamow peak: \( E_0 = 0.122 \mu^{1/3} (Z_1 Z_2 T_9)^{2/3} \) MeV: lower than the Coulomb barrier increases with \( T \)

\[ \Delta E_0 = 0.237 \mu^{1/6} (Z_1 Z_2)^{1/3} T_9^{5/6} \text{ MeV} \]

= Energy range where \( \sigma(E) \) must be known (\( T_9 = T \) in \( 10^9 \)K)
3. Reaction rates

Examples

<table>
<thead>
<tr>
<th>Reaction</th>
<th>T ($10^9$ K)</th>
<th>$E_0$ (MeV)</th>
<th>$\Delta E_0$ (MeV)</th>
<th>$E_{coul}$ (MeV)</th>
<th>$\sigma(E_0)/\sigma(E_{coul})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>d + p</td>
<td>0.015</td>
<td>0.006</td>
<td>0.007</td>
<td>0.3</td>
<td>$10^{-4}$</td>
</tr>
<tr>
<td>$^3$He + $^3$He</td>
<td>0.015</td>
<td>0.021</td>
<td>0.012</td>
<td>1.2</td>
<td>$10^{-13}$</td>
</tr>
<tr>
<td>$\alpha + ^{12}$C</td>
<td>0.2</td>
<td>0.3</td>
<td>0.17</td>
<td>3</td>
<td>$10^{-11}$</td>
</tr>
<tr>
<td>$^{12}$C + $^{12}$C</td>
<td>1</td>
<td>2.4</td>
<td>1.05</td>
<td>7</td>
<td>$10^{-10}$</td>
</tr>
</tbody>
</table>

$\Rightarrow$ Always $E_0 \ll E_{coul}$ (coulomb barrier)

$\Rightarrow$ Very low cross sections at stellar temperatures
3. Reaction rates

3. Non-resonant reaction rates

- Approximation: Taylor expansion about the minimum $E = E_0$: $2\pi\eta + E/k_BT \approx c_0 + \left(\frac{E-E_0}{2\Delta E_0}\right)^2$

Then $<\sigma v> \approx \left(\frac{8}{\pi\mu m_N(k_BT)^3}\right)^{1/2} \exp\left(-3\frac{E_0}{k_BT}\right) \int S(E) \exp\left(-\left(\frac{E-E_0}{2\Delta E_0}\right)^2\right) dE$

- $S(E)$ is assumed constant ($= S(E_0)$) in the Gamow peak

$$\rightarrow <\sigma v> \sim S(E_0) \exp\left(-3\frac{E_0}{k_BT}\right)/T^{2/3}, \text{with } E_0 = 0.122\mu^{1/3}(Z_1Z_2T_9)^{2/3} \text{ MeV}$$

![Graph of $\text{He}(\alpha,\gamma)^7\text{Be}$ reaction rate vs temperature](image)
3. Reaction rates

4. Resonant reaction rates

- General definition: \( N_A < \sigma v > = N_A \int \sigma(E)\nu N(E,T)dE \)

Here \( \sigma(E) \) is given by the Breit-Wigner approximation

\[
\sigma(E) \approx \frac{\pi}{k^2} \frac{(2J_R + 1)}{(2I_1 + 1)(2I_2 + 1)} \frac{\Gamma_1(E)\Gamma_2(E)}{(E_R - E)^2 + \Gamma^2/4}
\]

- This provides

\[
< \sigma v >_R = \left( \frac{2\pi}{\mu m_N k_B T} \right)^{3/2} \frac{\hbar^2 \omega \gamma \exp \left(-\frac{E_R}{k_B T}\right)}{2J_R + 1}
\]

\[
\omega \gamma = \frac{2J_R + 1}{(2I_1 + 1)(2I_2 + 1)} \frac{\Gamma_1 \Gamma_2}{\Gamma_1 + \Gamma_2}
\]

- \( \omega \gamma \) = resonance « strength »
- \( \Gamma_1, \Gamma_2 \) = partial widths in the entrance and exit channels
- For a reaction \((p,\gamma)\) : \( \Gamma_\gamma << \Gamma_p \rightarrow \omega \gamma \sim \Gamma_\gamma \)

- Valid for capture and transfer
- Rate strongly depends on the resonance energy

⇒ In general: competition between resonant and non-resonant contributions
3. Reaction rates

Tail contribution: for a given resonance

For a resonance: \( < \sigma v > \sim \int S(E) \exp(-2\pi \eta - E/k_B T) \, dE \)

- Non resonant: \( S(E) \approx S_0 \): 1 maximum at \( E = E_0 \)
- Resonant: \( S(E) = \text{BW} \): 2 maxima at \( E = E_R \) does not depend on \( T \)
  \( E = E_0 \): depends on \( T \)

\( \rightarrow \) 2 contributions to the rate: \( N_A \, < \sigma v > \approx N_A \, < \sigma v >_R + N_A \, < \sigma v >_T \)

\( < \sigma v >_R = \left( \frac{2\pi}{\mu m_N k_B T} \right)^{3/2} \hbar^2 \omega \gamma \exp \left( - \frac{E_R}{k_B T} \right) \)

\( < \sigma v >_T \sim S(E_0) \exp\left( -3 \frac{E_0}{k_B T} \right) / T^{2/3} \), with \( S(E_0) \sim \frac{\Gamma_1(E_0) \Gamma_2(E_0)}{(E_R-E_0)^2+\Gamma^2/4} \)

- Both contributions depend on temperature: in most cases 1 is dominant
- « Critical temperature »: when \( E_0 = E_R \) \( \rightarrow \) separation not valid
3. Reaction rates

Example $^{12}\text{C}(p,\gamma)^{13}\text{N}$: $E_R = 0.42$ MeV

Integrand $S(E)\exp(-2\pi\eta - E/k_BT)$

Above $T_9 \approx 0.3$: « resonant » contribution is dominant
requires $E_R, \omega\gamma$ only (no individual partial widths)
weakly depends on $E_R$: $\exp(-E_R/k_BT)$

Below $T_9 \approx 0.2$: $E_0 \ll E_R$: « tail » contribution is dominant
requires both widths
weakly depends on $E_R$: $1/((E_R - E_0)^2 + \Gamma^2/4)$
4. General scattering theory

1. Different models
2. Potential/optical model
3. Scattering amplitude and cross section (elastic scattering)
4. General scattering theory

Scheme of the collision (elastic scattering)

Before collision

After collision

Center-of-mass system
4. General scattering theory

1. Different models

Schrödinger equation: \( H\Psi(r_1, r_2, \ldots r_A) = E\Psi(r_1, r_2, \ldots r_A) \) with \( E > 0 \): scattering states

- A-body equation (microscopic models)
  \[
  H = \sum_i t_i + \frac{1}{2} \sum_{i,j} v_{ij}(r_i - r_j)
  \]
  \( v_{ij} \) = nucleon-nucleon interaction

- Optical model: internal structure of the nuclei is neglected
  the particles interact by a nucleus-nucleus potential
  absorption simulated by the imaginary part = optical potential
  \[
  H\Psi(r) = \left( -\frac{\hbar^2}{2\mu} \Delta + V(r) \right)\Psi(r) = E\Psi(r)
  \]

- Additional assumptions: elastic scattering
  no Coulomb interaction
  spins zero
2. Potential/Optical model

Two contributions to the nucleus-nucleus potential: nuclear $V_N(r)$ and Coulomb $V_C(r)$

Typical nuclear potential: $V_N(r)$ (short range, attractive)

- examples: Gaussian
  \[ V_N(r) = -V_0 \exp\left(-\left(\frac{r}{r_0}\right)^2\right) \]
  
  Woods-Saxon:
  \[ V_N(r) = -\frac{V_0}{1 + \exp\left(\frac{r-r_0}{a}\right)} \]

- Real at low energies
- parameters are fitted to experiment
- no analytical solution of the Schrödinger equation

Woods-Saxon potential

$r_0 =$ range (~sum of the radii)

$a =$ diffuseness (~0.5 fm)

Figure: $V_0=50$ MeV, $r_0=5$ fm, $a = 0.5$ fm
4. General scattering theory

**Coulomb potential:** long range, repulsive

- « point-point » potential : \( V_C(r) = \frac{Z_1Z_2e^2}{r} \)

- « point-sphere » potential : (radius \( R_C \))
  \[
  V_C(r) = \frac{Z_1Z_2e^2}{r} \text{ for } r \geq R_C
  \]
  \[
  V_C(r) = \frac{Z_1Z_2e^2}{2R_C} \left( 3 - \left( \frac{r}{R_C} \right)^2 \right) \text{ for } r \leq R_C
  \]

**Total potential** : \( V(r) = V_N(r) + V_C(r) \): presents a maximum at the Coulomb barrier

- radius \( r = R_B \)
- height \( V(R_B) = E_B \)
3. Scattering amplitude and cross section

\[ H \Psi(r) = \left( -\frac{\hbar^2}{2\mu} \Delta + V(r) \right) \Psi(r) = E \Psi(r) \]

with \( \Psi(r) = \Phi(r) + \Psi_{\text{scatt}}(r) \) (\( \Phi(r) \) corresponds to \( V(r)=0 \))

At large distances: \( \Psi(r) \to A \left( e^{i k \cdot r} + f(\theta) \frac{e^{i k r}}{r} \right) \) (with \( z \) along the beam axis)

where: \( k = \) wave number: \( k^2 = 2 \mu E / \hbar^2 \)
\( A = \) amplitude (scattering wave function is not normalized to unity)
\( f(\theta) = \) scattering amplitude (length)

4. General scattering theory
4. General scattering theory

Cross section: \[ \frac{d\sigma}{d\Omega} = |f(\theta)|^2, \quad \sigma = \int \frac{d\sigma}{d\Omega} d\Omega \]

- Cross section obtained from the asymptotic part of the wave function
  
  **General problem for scattering states**: the wave function must be known up to large distances

- “Direct” problem: determine \( \sigma \) from the potential

- “Inverse” problem: determine the potential \( V \) from \( \sigma \)

- **Angular distribution**: \( E \) fixed, \( \theta \) variable

- **Excitation function**: \( \theta \) variable, \( E \) fixed,
4. General scattering theory

Main issue: determining the scattering amplitude $f(\theta)$ (and wave function $\Psi(r)$)

At low energies: partial wave expansion: $\Psi(r) = \sum_{lm} \Psi_l(r)Y_{l}^{m}(\theta, \phi)$

- Scattering wave function necessary to compute cross sections
- Must be determined for each partial wave $l$
- Main interest: few partial waves at low energies

\[ V \]

\[ \ell > 0 \]

\[ \ell = 0 \]

Centrifugal term: $\frac{\hbar^2}{2\mu} \frac{\ell(\ell+1)}{r^2}$

Partial-wave expansion
- Only a few partial waves contribute
- Effect more important for nucleon-nucleus: $\mu \approx 1$
- Strongest for neutron: no barrier for $\ell = 0$. 
4. General scattering theory

4. Phase-shift method

- Goal: solving the Schrödinger equation

\[
\left( -\frac{\hbar^2}{2\mu} \Delta + V(r) \right) \Psi(r) = E \Psi(r)
\]

with a partial-wave expansion

\[
\Psi(r) = \sum_{\ell,m} \frac{u_{\ell}(r)}{r} Y_{\ell}^{m}(\Omega_{r}) Y_{\ell}^{m*}(\Omega_{k})
\]

- Simplifying assumptions
  - neutral systems (no Coulomb interaction)
  - spins zero
  - single-channel calculations \(\rightarrow\) elastic scattering
4. General scattering theory

- The wave function is expanded as

\[ \Psi(r) = \sum_{\ell,m} \frac{u_\ell(r)}{r} Y_\ell^m(\Omega_r) Y_\ell^m*(\Omega_k) \]

- This provides the Schrödinger equation for each partial wave \((\Omega_k = 0 \rightarrow m = 0)\)

\[ -\frac{\hbar^2}{2\mu} \left( \frac{d^2}{dr^2} - \frac{\ell(\ell + 1)}{r^2} \right) u_\ell + V(r) u_\ell = E u_\ell \]

- Large distances: \(r \rightarrow \infty, V(r) \rightarrow 0\)

\[ u_\ell'' - \frac{\ell(\ell + 1)}{r^2} u_\ell + k^2 u_\ell = 0 \]

Bessel equation \(\rightarrow u_\ell(r) = rj_\ell(kr), rn_\ell(kr)\)

- Remarks
  - must be solved for all \(\ell\) values
  - at low energies: few partial waves in the expansion
  - at small \(r\): \(u_\ell(r) \rightarrow r^{\ell+1}\)
4. General scattering theory

For small $x$: 
\[ j_l(x) \rightarrow \frac{x^l}{(2l+1)!!} \]
\[ n_l(x) \rightarrow -\frac{x^l}{(2l-1)!!} \]

For large $x$: 
\[ j_l(x) \rightarrow \frac{1}{x} \sin(x - l\pi/2) \]
\[ n_l(x) \rightarrow -\frac{1}{x} \cos(x - l\pi/2) \]

Examples: 
\[ j_0(x) = \frac{\sin x}{x}, \quad n_0(x) = -\frac{\cos x}{x} \]

At large distances: $u_\ell(r)$ is a linear combination of $rj_\ell(kr)$ and $rn_\ell(kr)$

\[ u_\ell(r) \rightarrow C_\ell \ r \ (j_\ell(kr) - \tan \delta_\ell \times n_\ell(kr)) \]

With $\delta_\ell = \text{phase shift}$ (provides information about the potential): If $V=0 \rightarrow \delta_\ell = 0$
Derivation of the elastic cross section

- Identify the asymptotic behaviours
  \[ \Psi(r) \to A \left( e^{i k \cdot r} + f(\theta) \frac{e^{i k r}}{r} \right) \]
  \[ \Psi(r) \to \sum_\ell C_\ell (j_\ell(kr) - \tan \delta_\ell \times n_\ell(kr)) Y_\ell^0(\Omega_r) \sqrt{\frac{2\ell + 1}{4\pi}} \]

- Provides coefficients \( C_\ell \) and scattering amplitude \( f(\theta) \) (elastic scattering)
  \[ f(\theta, E) = \frac{1}{2ik} \sum_{\ell=0}^{\infty} (2\ell + 1)(\exp(2i\delta_\ell(E)) - 1) P_\ell(\cos \theta) \]
  \[ \frac{d\sigma(\theta,E)}{d\Omega} = |f(\theta,E)|^2 \]

- Integrated cross section (neutral systems only)
  \[ \sigma = \frac{\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell + 1) \sin^2 \delta_\ell \]
  \[ |f(\theta,E)|^2 \]

- In practice, the summation over \( \ell \) is limited to some \( \ell_{\text{max}} \)
4. General scattering theory

\[ \frac{d\sigma(\theta, E)}{d\Omega} = |f(\theta, E)|^2 \text{ with } f(\theta, E) = \frac{1}{2ik} \sum_{\ell=0}^{\infty} (2\ell + 1)(\exp(2i\delta_\ell(E)) - 1) P_\ell(\cos \theta) \]

⇒ **factorization** of the dependences in \( E \) and \( \theta \)
- low energies: small number of \( \ell \) values (\( \delta_\ell \to 0 \) when \( \ell \) increases)
- high energies: large number (\( \to \) alternative methods)

**General properties of the phase shifts**

1. The phase shift (and all derivatives) are continuous functions of \( E \)
2. The phase shift is known within \( n\pi \): \( \exp 2i\delta = \exp(2i(\delta + n\pi)) \)
3. Levinson theorem
   - \( \delta_\ell(E = 0) \) is arbitrary
   - \( \delta_\ell(0) - \delta_\ell(\infty) = N\pi \), where \( N \) is the number of bound states in partial wave \( \ell \)
   - Example: \( p+n \), \( \ell = 0: \delta_0(0) - \delta_0(\infty) = \pi \) (bound deuteron)
     \( \ell = 1: \delta_1(0) - \delta_1(\infty) = 0 \) (no bound state for \( \ell = 1 \))
4. General scattering theory

- Example: hard sphere (radius a)
- Continuity at \( r = a \rightarrow j_\ell(ka) - \tan \delta_\ell \times n_\ell(ka) = 0 \)
  \[ \tan \delta_\ell = \frac{j_\ell(ka)}{n_\ell(ka)} \]
  \[ \delta_0 = -ka \]

At low energies: \( \delta_\ell(E) \rightarrow -\frac{(ka)^{2\ell+1}}{(2\ell+1)!!(2\ell-1)!!}, \) in general: \( \delta_\ell(E) \sim k^{2\ell+1} \)

\( \Rightarrow \) Strong difference between \( \ell = 0 \) (no barrier) et \( \ell \neq 0 \) (centrifugal barrier)
(typical to neutron-induced reactions)
example: $\alpha+n$ phase shift $\ell = 0$

consistent with the hard sphere ($a \sim 2.2$ fm)
5. Resonances

Resonances: \( \delta_R(E) \approx \text{atan} \left( \frac{\Gamma}{2(E_R - E)} \right) \) = Breit-Wigner approximation

\( E_R = \) resonance energy
\( \Gamma = \) resonance width: related to the lifetime \( \Gamma \tau = h \)

- Narrow resonance: \( \Gamma \) small, \( \tau \) large
- Broad resonance: \( \Gamma \) large, \( \tau \) small
- Bound states: \( \Gamma = 0, E_R < 0 \)
4. General scattering theory

Cross section (for neutrons)

\[ \sigma(E) = \frac{\pi}{k^2} \sum_\ell (2\ell + 1) |\exp(2i\delta_\ell) - 1|^2 \quad \text{maximum for } \delta = \frac{\pi}{2} \]

Near the resonance: \( \sigma(E) \approx \frac{4\pi}{k^2} (2\ell_R + 1) \frac{\Gamma^2/4}{(E_R-E)^2 + \Gamma^2/4} \), where \( \ell_R = \text{resonant partial wave} \)

In practice:
- Peak not symmetric (\( \Gamma \) depends on \( E \))
- « Background » neglected (other \( \ell \) values)
- Differences with respect to Breit-Wigner
4. General scattering theory

Example: n$^{12}$C

Comparison of 2 typical times:

a. Lifetime of the resonance: $\tau_R = \hbar/\Gamma \approx \frac{197}{3.10^{23} \times 6.10^{-3}} \approx 1.1 \times 10^{-19} \text{s}$

b. Interaction time without resonance: $\tau_{NR} = \frac{d}{\nu} \approx 5.2 \times 10^{-22} \text{s} \Rightarrow \tau_{NR} \ll \tau_R$

$E_{cm} = 1.92 \text{ MeV}$
$\Gamma = 6 \text{ keV}$
4. General scattering theory

**Narrow resonances**
- Small particle width
- Long lifetime
- Can be approximately treated by *neglecting the asymptotic behaviour of the wave function*

Proton width = 32 keV

→ Can be described in a bound-state approximation
4. General scattering theory

Broad resonances
- Large particle width
- Short lifetime
- *asymptotic behaviour of the wave function is important*
  → rigorous scattering theory
  → bound-state model complemented by other tools (complex scaling, etc.)

\[ G = 120 \text{ keV} \]

\[ G = 1990 \text{ keV} \]

very broad resonance: \( \Gamma = 1990 \text{ keV} \)

ground state unstable: \( \Gamma = 120 \text{ keV} \)
5. Generalizations

- Extension to charged systems
- Numerical calculation
- Optical model (high energies → absorption)
- Extension to multichannel problems
5. Generalizations

Generalization 1: charged systems

\[ E \gg E_B: \text{ weak coulomb effects (} V \text{ negligible with respect to } E \text{)} \]
\[ E < E_B: \text{ strong coulomb effects (ex: nuclear astrophysics)} \]
4. Generalizations

A. Asymptotic behaviour

**Neutral systems**

\[
\left( -\frac{\hbar^2}{2\mu} \Delta + V_N(r) - E \right) \Psi(r) = 0
\]

\[
\Psi(r) \rightarrow \exp(i \mathbf{k} \cdot \mathbf{r}) + f(\theta) \frac{\exp(i k r)}{r}
\]

**Charged systems**

\[
\left( -\frac{\hbar^2}{2\mu} \Delta + V_N(r) + \frac{Z_1 Z_2 e^2}{r} - E \right) \Psi(r) = 0
\]

\[
\Psi(r) \rightarrow \exp(i \mathbf{k} \cdot \mathbf{r} + i \eta \ln(\mathbf{k} \cdot \mathbf{r} - kr))
\]

\[
+ f(\theta) \frac{\exp(i (kr - \eta \ln 2kr))}{r}
\]

\[
\eta = \frac{Z_1 Z_2 e^2}{\hbar v}
\]

- Sommerfeld parameter
- « measurement » of coulomb effects
- Increases at low energies
- Decreases at high energies
5. Generalizations

B. Phase shifts with the coulomb potential

Neutral system:
\[ \left( \frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} + k^2 \right) R_\ell = 0 \]

Bessel equation: solutions \( j_\ell(kr), n_\ell(kr) \)

Charged system:
\[ \left( \frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} - 2 \frac{\eta k}{r} + k^2 \right) R_\ell = 0: \]
Coulomb equation: solutions \( F_\ell(\eta, kr), G_\ell(\eta, kr) \)
5. Generalizations

• Incoming and outgoing functions (complex)

\[ I_\ell(\eta, x) = G_\ell(\eta, x) - iF_\ell(\eta, x) \to e^{-i(x-\frac{\ell \pi}{2} - \eta \ln 2x + \sigma_\ell)}: \text{incoming wave} \]

\[ O_\ell(\eta, x) = G_\ell(\eta, x) + iF_\ell(\eta, x) \to e^{i(x-\frac{\ell \pi}{2} - \eta \ln 2x + \sigma_\ell)}: \text{outgoing wave} \]

• Phase-shift definition
  
  o neutral systems: \( R_\ell(r) \to rA(j_\ell(kr) - \tan \delta_\ell n_\ell(kr)) \)
  
  o charged systems: \( R_\ell(r) \to A(F_\ell(\eta, kr) + \tan \delta_\ell G_\ell(\eta, kr)) \)
  \[ \to B(\cos \delta_\ell F_\ell(\eta, kr) + \sin \delta_\ell G_\ell(\eta, kr)) \]
  \[ \to C(I_\ell(\eta, kr) - U_\ell O_\ell(\eta, kr)) \]

  3 equivalent definitions (amplitude is different)

Collision matrix (=scattering matrix)

\[ U_\ell = e^{2i\delta_\ell} : \text{module } |U_\ell| = 1 \]
5. Generalizations

Example: hard-sphere potential

\[ V(r) = \begin{cases} \frac{Z_1 Z_2 e^2}{r} & \text{for } r > a \\ \infty & \text{for } r < a \end{cases} \]

phase shift: \( \tan \delta_\ell = -\frac{F_\ell(\eta, k\alpha)}{G_\ell(\eta, k\alpha)} \)

\[ k (fm^{-1}) \]

\[ \begin{array}{c}
\eta=0 \\
\ell=1 \\
\eta=2 \\
\ell=2 \\
\end{array} \]

\[ a=4 \text{ fm} \]
5. Generalizations

C. Rutherford cross section
For a Coulomb potential \( V_N = 0 \):

- scattering amplitude: \( f_c(\theta) = -\frac{\eta}{2k \sin^2 \theta/2} e^{2i(\sigma_0 - \eta \ln \sin \theta/2)} \)
- Coulomb phase shift for \( \ell = 0 \): \( \sigma_0 = \arg \Gamma(1 + i\eta) \)

We get the Rutherford cross section:

\[
\frac{d\sigma_C}{d\Omega} = |f_c(\theta)|^2 = \left( \frac{Z_1 Z_2 e^2}{4E \sin^2 \theta/2} \right)^2
\]

- Increases at low energies
- Diverges at \( \theta = 0 \) \( \rightarrow \) no integrated cross section
5. Generalizations

D. Cross sections with nuclear and Coulomb potentials

• The general definitions

\[
f(\theta) = \frac{1}{2ik} \sum_{\ell=0}^{\infty} (2\ell + 1)(\exp(2i\delta_{\ell}) - 1) P_{\ell}(\cos \theta)
\]

\[
\frac{d\sigma}{d\Omega} = |f(\theta)|^2
\]

are still valid

• Problem: very slow convergence with \( \ell \)
  \( \rightarrow \) separation of the nuclear and coulomb phase shifts

\[
\delta_{\ell} = \delta_{\ell}^N + \sigma_{\ell}
\]

\[
\sigma_{\ell} = \text{arg}\Gamma(1 + \ell + i\eta)
\]

• Scattering amplitude \( f(\theta) \) written as \( f(\theta) = f^C(\theta) + f^N(\theta) \)
  \( f^C(\theta) = \frac{1}{2ik} \sum_{\ell=0}^{\infty} (2\ell + 1)(\exp(2i\sigma_{\ell}) - 1) P_{\ell}(\cos \theta) = -\frac{\eta}{2k \sin^2 \theta / 2} e^{2i(\sigma_0 - \eta \ln \sin \theta / 2)} \)
  \( \rightarrow \) analytical

• \( f^N(\theta) = \frac{1}{2ik} \sum_{\ell=0}^{\infty} (2\ell + 1) \exp(2i\sigma_{\ell}) (\exp(2i\delta_{\ell}^N) - 1) P_{\ell}(\cos \theta) \)
  \( \rightarrow \) converges rapidly
5. Generalizations

Total cross section: \( \frac{d\sigma}{d\Omega} = |f(\theta)|^2 = |f^C(\theta) + f^N(\theta)|^2 \)

- Nuclear term dominant at 180°
- Coulomb term coulombien dominant at small angles \( \rightarrow \) used to normalize experiments
- Coulomb amplitude strongly depends on the angle \( \rightarrow \) \( \frac{d\sigma}{d\Omega} \) / \( \frac{d\sigma_C}{d\Omega} \)
- Integrated cross section \( \int \frac{d\sigma}{d\Omega} \, d\Omega \) is not defined

**System \(^6\text{Li} + ^{58}\text{Ni}\)**

- \( E_{cm} = \frac{58}{64} E_{lab} \)
- Coulomb barrier \( E_B \sim \frac{3 \times 28 \times 1.44}{7} \sim 17 \text{ MeV} \)
- Below the barrier: \( \sigma \sim \sigma_C \)
- Above \( E_B \): \( \sigma \) is different from \( \sigma_C \)
Generalization 2: numerical calculation

For some potentials: analytic solution of the Schrödinger equation

In general: no analytical solution $\Rightarrow$ numerical approach

$$-\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} u_\ell(r) + (V(r) - E) u_\ell(r) = 0$$

with:

$$V(r) = V_N(r) + \frac{Z_1 Z_2 e^2}{r} + \frac{\hbar^2 \ell (\ell + 1)}{2\mu r^2}$$

$$u_\ell(r) \rightarrow F_\ell(kr, \eta) \cos \delta_\ell + G_\ell(kr, \eta) \sin \delta_\ell$$

Numerical solution: discretization $N$ points, with mesh size $h$

- $u_\ell(0) = 0$
- $u_\ell(h) = 1$ (or any constant)
- $u_\ell(2h)$ is determined numerically from $u_\ell(0)$ and $u_\ell(h)$ (Numerov algorithm)
- $u_\ell(3h), ... u_\ell(Nh)$
- for large $r$: matching to the asymptotic behaviour $\Rightarrow$ phase shift

Bound states: same idea (but energy is unknown)
5. Generalizations

Example: $\alpha+\alpha$

Experimental spectrum of $^8$Be

- **$4^+$**
  - $E \approx 11$ MeV
  - $\Gamma \approx 3.5$ MeV

- **$2^+$**
  - $E \approx 3$ MeV
  - $\Gamma \approx 1.5$ MeV

- **$0^+$**
  - $E = 0.09$ MeV
  - $\Gamma = 6$ eV

Potential: $V_N(r) = -122.3 \exp(-r/2.13)^2$

Experimental phase shifts

- $l=0$
- $l=2$
- $l=4$

Potential: $V_B \approx 1$ MeV

Experimental spectrum of $^8$Be
5. Generalizations

\(\alpha + \alpha\) wave function for \(\ell = 0\)

- \(E = 0.2\) MeV
  - \(E < E_B\)
  - Small amplitude for \(r\) small

- \(E = 1\) MeV
  - \(E \approx E_B\)
5. Generalizations

Generalization 3: complex potentials $V = V_R + iW$

Goal: to simulate absorption channels

High energies:
- many open channels
- strong absorption
- potential model extended to complex potentials (« optical »)

Phase shift is complex: $\delta = \delta_R + i\delta_I$

collision matrix: $U = \exp(2i\delta) = \eta \exp(2i\delta_R)$
where $\eta = \exp(-2\delta_I) < 1$

Elastic cross section

$$\frac{d\sigma}{d\Omega} = \frac{1}{4k^2} \left| \sum_{\ell} (2\ell + 1) (\eta_\ell \exp(2i\delta_\ell) - 1) P_\ell(\cos \theta) \right|^2$$

Reaction cross section:

$$\sigma = \frac{\pi}{k^2} \sum_{\ell} (2\ell + 1) (1 - \eta_\ell^2)$$
5. Generalizations

In astrophysics, optical potentials are used to compute fusion cross sections.

Fusion cross section: includes many channels

Example: $^{12}\text{C} + ^{12}\text{C}$: Essentially $^{20}\text{Ne} + ^{12}\text{C}$, $^{23}\text{Na} + p$, $^{23}\text{Mg} + n$ channels → absorption simulated by a complex potential $V = V_R + iW$

experimental cross section
Satkowiak et al. PRC 26 (1982) 2027
5. Generalizations

Generalization 4: system with spins (multichannel)

- Allows to deal with inelastic, transfer, etc..
- Phase shift (single-channel) → collision (scattering) matrix

A. Quantum numbers

- Good quantum numbers: total angular momentum $J$ and parity $\pi$
- Additional indices
  - Channel $\alpha$ defined by 2 nuclei with spins $I_1, I_2$ and parities $\pi_1, \pi_2$
  - Channel spin $I = I_1 + I_2$
  - Relative angular momentum $\ell$
    
    \[
    J = I + \ell \\
    \pi = \pi_1 \pi_2 (-1)^\ell
    \]

Examples:

1) $\alpha+n: I_1 = 0, I_2 = 1/2 \rightarrow I = 1/2, \ell = |J - 1/2|$ or $J + 1/2$: channel number = 1

2) p+n: $I_1 = I_2 = 1/2 \rightarrow I = 0$ or 1: channel number depends on $J$
5. Generalizations

3) Reaction $^6\text{Li} + p \rightarrow ^3\text{He} + \alpha$

- channel 1: $^6\text{Li} + p$, spin($^6\text{Li}$) $I_1=1^+$, spin(p) $I_2=1/2^+$
- channel 2: $^3\text{He} + \alpha$, spin ($^3\text{He}$)$=1/2^+$, spin($\alpha$)$=0^+$

<table>
<thead>
<tr>
<th>$J\pi$</th>
<th>channel $\alpha = 1$</th>
<th>channel $\alpha = 2$</th>
<th>$\alpha I \ell$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1/2^+$</td>
<td>$I = 1/2, \ell = 0$</td>
<td>$I = 1/2, \ell = 0$</td>
<td>3 values</td>
</tr>
<tr>
<td></td>
<td>$I = 3/2, \ell = 2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$1/2^-$</td>
<td>$I = 1/2, \ell = 1$</td>
<td>$I = 1/2, \ell = 1$</td>
<td>3 values</td>
</tr>
<tr>
<td></td>
<td>$I = 3/2, \ell = 1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$3/2^+$</td>
<td>$I = 1/2, \ell = 2$</td>
<td>$I = 1/2, \ell = 2$</td>
<td>4 values</td>
</tr>
<tr>
<td></td>
<td>$I = 3/2, \ell = 0,2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$3/2^-$</td>
<td>$I = 1/2, \ell = 1$</td>
<td>$I = 1/2, \ell = 1$</td>
<td>4 values</td>
</tr>
<tr>
<td></td>
<td>$I = 3/2, \ell = 1,3$</td>
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</tr>
</tbody>
</table>

→ Size of the collision matrix is: 3x3 or 4x4
C. Cross sections in a multichannel formalism

One channel: 
\[ \frac{d\sigma}{d\Omega} = |f(\theta)|^2 \]
\[ f(\theta) = \frac{1}{2ik} \sum_{\ell=0}^{\infty} (2\ell + 1)(\exp(2i\delta_{\ell}) - 1) P_{\ell}(\cos \theta) \]

Multi channel: 
\[ \frac{d\sigma}{d\Omega}(\alpha \rightarrow \alpha') = \sum_{K_1K_2K'_1K'_2} \left| f_{K_1K_2K'_1K'_2}(\theta) \right|^2 \]
\[ f_{K_1K_2K'_1K'_2}(\theta) = \sum_{J,\pi} \sum_{l, l'} Y_{l'}(\theta, 0) \]
\[ U_{\alpha l \ell, \alpha' l' \ell'} J^\pi \]

Collision matrix
- generalization of d: \[ U_{ij} = \eta_{ij} \exp(2i\delta_{ij}) \]
- determines the cross section

With: \( K_1K_2 \) = spin orientations in the entrance channel
\( K'_1K'_2 \) = spin orientations in the exit channel
6. Models used for nuclear reactions in astrophysics
6. Models used in nuclear astrophysics (for reactions)

Theoretical methods: Many different cases → no “unique” model!

<table>
<thead>
<tr>
<th>Model</th>
<th>Applicable to</th>
<th>Comments</th>
</tr>
</thead>
</table>
| **Potential/optical model** | Capture, Fusion       | • Internal structure neglected  
                              • Antisymmetrization approximated                                    |
| **R-matrix**              | Capture, Transfer     | • No explicit wave functions  
                              • Physics simulated by some parameters                                |
| **DWBA**                  | Transfer              | • Perturbation method  
                              • Wave functions in the entrance and exit channels                   |
| **Microscopic models**    | Capture, Transfer     | • Based on a nucleon-nucleon interaction  
                              • A-nucleon problems  
                              • Predictive power                                                   |
| **Hauser-Feshbach**       | Capture, Transfer     | • Statistical model                                                     |
| **Shell model**           | Capture, Capture      | • Only gamma widths                                                     |
7. Radiative capture in the potential model
7. Radiative capture in the potential model

Potential model: two structureless particles (=optical model, without imaginary part)

- Calculations are simple
- Physics of the problem is identical in other methods
- Spins are neglected

- \( R_{cm} \)=center of mass, \( r \)=relative coordinate

\[
\begin{align*}
    r_1 &= R_{cm} - \frac{A_2}{A} r \\
    r_2 &= R_{cm} + \frac{A_1}{A} r
\end{align*}
\]

- Initial wave function: \( \Psi^{\ell i m_i}(r) = \frac{1}{r} u_{\ell i}(r) Y_{\ell i}^{m_i}(\Omega) \), energy \( E^{\ell i} \)=scattering energy \( E \)

- Final wave function: \( \Psi^{\ell f m_f}(r) = \frac{1}{r} u_{\ell f}(r) Y_{\ell f}^{m_f}(\Omega) \), energy \( E^{\ell f} \)

The radial wave functions are given by:

\[
- \frac{\hbar^2}{2\mu} \left( \frac{d^2}{dr^2} - \frac{\ell(\ell + 1)}{r^2} \right) u_\ell + V(r)u_\ell = E^\ell u_\ell
\]
7. Radiative capture in the potential model

- Schrödinger equation: 
  \[-\frac{\hbar^2}{2\mu} \left( \frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} \right) u_\ell + V(r)u_\ell = E^\ell u_\ell\]

- Typical potentials:
  - coulomb = point-sphere
  - nuclear: Woods-Saxon, Gaussian
    parameters adjusted on important properties (bound-state energy, phase shifts, etc.)

- Potentials can be different in the initial and final states

- Wave functions computed numerically (Numerov algorithm)

- Limitations
  - initial (scattering state): must reproduce resonances (if any)
  - final (bound) state: must have a A+B structure

\[\text{initial state } E > 0\]
\[\text{final state } E < 0\]
7. Radiative capture in the potential model

Some typical examples

\[ \alpha + ^3\text{H} \text{ structure} \]

\[ n + ^{7}\text{Li} \text{ structure} \]

\[ n + ^{16}\text{O} \text{ structure} \]

\[ ^7\text{Li} \]

Problem more and more important when the level density increases

→ in practice: limited to low-level densities (light nuclei or nuclei close to the drip lines)
7. Radiative capture in the potential model

• Electric operator for two particles:

\[ \mathcal{M}_{\mu}^{E\lambda} = e \left( Z_1 | \mathbf{r}_1 - \mathbf{R}_{cm} |^{\lambda} Y_\lambda^\mu (\Omega_{r_1-R_{cm}}) + Z_2 | \mathbf{r}_2 - \mathbf{R}_{cm} |^{\lambda} Y_\lambda^\mu (\Omega_{r_2-R_{cm}}) \right) \]

which provides

\[ \mathcal{M}_{\mu}^{E\lambda} = e \left[ Z_1 \left( -\frac{A_2}{A} \right)^{\lambda} + Z_2 \left( \frac{A_1}{A} \right)^{\lambda} \right] r^{\lambda} Y_\lambda^\mu (\Omega_r) = e Z_{eff} r^{\lambda} Y_\lambda^\mu (\Omega_r) \]

• Matrix elements needed for electromagnetic transitions

\[ < \Psi_{J_f m_f} | \mathcal{M}_{\mu}^{E\lambda} | \Psi_{J_i m_i} > = e Z_{eff} < Y_{J_f m_f}^\mu | Y_{J_i}^m_i > \int_0^\infty u_{J_i} (r) u_{J_f} (r) r^{\lambda} dr \]

• Reduced matrix elements:

\[ < \Psi_{J_f} \| \mathcal{M}^{E\lambda} \| \Psi_{J_i} > = e Z_{eff} < J_f 0 \lambda 0 | J_i 0 > \times \left( \frac{(2J_i+1)(2\lambda+1)}{4\pi(2J_f+1)} \right)^{1/2} \int_0^\infty u_{J_i} (r) u_{J_f} (r) r^{\lambda} dr \]

\[ \rightarrow \text{simple one-dimensional integrals} \]
7. Radiative capture in the potential model

**Assumptions:**
- spins zero: \( \ell_i = J_i, \ell_f = J_f \)
- given values of \( J_i, J_f, \lambda \)

**Integrated cross section**

\[
\sigma_\lambda(E) = \frac{8\pi e^2}{k^2 \hbar c} Z_{eff}^2 k^2 \gamma F(\lambda, J_i, J_f) \left| \int_0^\infty u_{J_i}(r, E) u_{J_f}(r) r^\lambda dr \right|^2
\]

with

- \( Z_{eff} = Z_1 \left( -\frac{A_2}{A} \right)^\lambda + Z_2 \left( \frac{A_1}{A} \right)^\lambda \)
- \( F(\lambda, J_i, J_f) = \langle J_i \lambda 0 0 | J_f 0 \rangle (2J_i + 1) \frac{(\lambda+1)(2\lambda+1)}{\lambda(2\lambda+1)!^2} \)
- \( k_\gamma = \frac{E-E_f}{\hbar c} \)

**Normalization**

- final state (bound): normalized to unity \( u_J(r) \to CW(2k_B r) \to Cexp(-k_B r) \)
- initial state (continuum): \( u_J(r) \to F_j(kr) \cos \delta_j + G_j(kr) \sin \delta_j \)
Integrated vs differential cross sections

- **Total (integrated) cross section:**
  \[
  \sigma(E) = \sum_{\lambda} \sigma_{\lambda}(E)
  \]
  → no interference between the multipolarities

- **Differential cross section:**
  \[
  \frac{d\sigma}{d\theta} = \left| \sum_{\lambda} a_{\lambda}(E) P_{\lambda}(\theta) \right|^2
  \]
  - \( P_{\lambda}(\theta) \)= Legendre polynomial
  - \( a_{\lambda}(E) \) are complex, \( \sigma_{\lambda}(E) \sim |a_{\lambda}(E)|^2 \)
    → interference effects
    → angular distributions are necessary to separate the multipolarities
    → in general one multipolarity is dominant (not in \(^{12}\text{C}(\alpha,\gamma)^{16}\text{O}: \text{E1 and E2}\)
7. Radiative capture in the potential model

Example: $^{12}\text{C}(p,\gamma)^{13}\text{N}$

- First reaction of the CNO cycle
- Well known experimentally
- Presents a low energy resonance ($\ell = 0 \rightarrow J = 1/2^+$)

Potential: $V = -55.3 \exp\left(-\frac{r}{2.70}\right)^2$ (final state)
$-70.5 \exp\left(-\frac{r}{2.70}\right)^2$ (initial state)
7. Radiative capture in the potential model

Final state: \( J_f = 1/2^- \)
Initial state: \( \ell_i = 0 \rightarrow J_i = 1/2^+ \)

\( \Rightarrow \) E1 transition \( 1/2^+ \rightarrow 1/2^- \)

Final state: \( \mathbf{E} = -1.94 \text{ MeV} \)

Initial wave function: \( E_i = 100 \text{ keV} \)

Integrant \( E_i = 100 \text{ keV} \)

Integral \( E_i = 100 \text{ keV} \)
7. Radiative capture in the potential model

The calculation is repeated at all energies

![Graph showing the relationship between S and E (MeV)]

Necessity of a spectroscopic factor $S$

Assumption of the potential model: $^{13}\text{N}=^{12}\text{C}+\text{p}$

In reality $^{13}\text{N}=^{12}\text{C}+\text{p} \oplus ^{12}\text{C}^*+\text{p} \oplus ^{9}\text{Be}+\alpha \oplus \ldots$

$\Rightarrow$ to simulate the missing channels: $u_f(r)$ is replaced by $S^{1/2}u_f(r)$

$S=$spectroscopic factor

Other applications: $^7\text{Be}(p,\gamma)^8\text{B}$, $^3\text{He}(\alpha,\gamma)^7\text{Be}$, etc…
Two comments
1. Spectroscopic factors
2. Internal-external contributions

Comment 1: Spectroscopic factors

Theoretical definition
Nucleus C described in terms of $A=A_1+A_2$

- Wave function of $A$ ($A$ nucleons): $\Psi_A(r_1, r_2, ..., r_A)$
  - A coordinates, but independent of c.m. $\rightarrow (A-1)$ independent coordinates
- Wave function of $A_1$ ($A_1$ nucleons): $\Phi_{A_1}:(A_1-1)$ independent coordinates
- Wave function of $A_2$ ($A_2$ nucleons): $\Phi_{A_2}:(A_2-1)$ independent coordinates

- Overlap integral $I(r) = \langle \Psi_A | \Phi_{A_1} \Phi_{A_2} \rangle$
  - $\rightarrow$ integration over internal coordinates $A_1+A_2-2$
  - $\rightarrow$ depends on one coordinate: the relative distance $r$

- Spectroscopic factor $S = \int I(r)r^2dr$
  - Probes the wave function over the whole space
  - Depends on the definition of $\Psi_A$, $\Phi_{A_1}$, $\Phi_{A_2}$
  - Can be defined for excited states of $A_1$ and $A_2$ $\langle \Psi_A | \Phi_{A_1}^* \Phi_{A_2}^* \rangle$
7. Radiative capture in the potential model

In practice:

• Various uses and definitions of the spectroscopic factor
• In many cases the wave function is needed but not known
  \( \rightarrow \) approximation \( \Psi_A \approx S \Phi_{A_1} \Phi_{A_2} g(r) \) with \( g(r) \) =relative function (potential model)

• Examples:
  • Radiative capture \( A_1(A_2,\gamma)A \):  
    \[
    \Psi_f \approx \sqrt{S_f} \Phi_{A_1} \Phi_{A_2} g_f(r) \\
    \Psi_i \approx \Phi_{A_1} \Phi_{A_2} g_i(r)
    \]
    \( \rightarrow \) matrix element \( < \Psi_f | M_\lambda | \Psi_i > = \sqrt{S_f} \int g_f(r)g_i(r)r^\lambda \)
  \( \rightarrow \) \( \sigma_{exp} = S_f \sigma_{pot} \)

• DWBA: various wave functions are needed \( \rightarrow \) same approximation

• Problem: are the spectroscopic factors compatible with the theoretical definition?

• Answer:  
  if \( S \approx 1 \): yes (but uncertainties!)
  if \( S \ll 1 \): no (approximation \( \Psi_A \approx S \Phi_{A_1} \Phi_{A_2} g(r) \) very poor, other terms are more important)
7. Radiative capture in the potential model

**Comment 2: Internal-external contributions**

\[
\sigma_\lambda(E) = \frac{8\pi e^2}{k^2 \hbar c} Z_{\text{eff}}^2 k^2 \lambda + 1 F(\lambda, J_i, J_f) \left| \int_0^{\infty} u_{J_i}(r, E) u_{J_f}(r) r^\lambda dr \right|^2
\]

- initial scattering wave function \( u_j(r) \to F_j(kr) \cos \delta_j + G_j(kr) \sin \delta_j \)

- final bound-state wave function: binding energy \( E_B \)
  \[ u_f(r) \to C W_{-\eta_B, \ell + 1/2}(2k_B r), \quad W_{-\eta_B, \ell + 1/2}(x) \] is the Whittaker function

\[ W_{-\eta_B, \ell + 1/2}(2k_B r) \to \exp(-k_B r) \]

\( C = \text{Aymptotic Normalization Constant (ANC)} \)

with \( k_B = \sqrt{2\mu E_B / \hbar^2} \)
7. Radiative capture in the potential model

Internal and external components of the integrant

\[ M_{\text{int}} = \int_0^a u_{j_i}(r, E) u_{j_f}(r) r^\lambda dr \]

\[ M_{\text{ext}} = \int_a^\infty u_{j_i}(r, E) u_{j_f}(r) r^\lambda dr \]

\( a = \) typical radius (\( \sim \) range of the nuclear interaction)
Capture cross section: \( \sigma \sim |M_{\text{int}} + M_{\text{ext}}|^2 \)
7. Radiative capture in the potential model

Capture cross section: \( \sigma \sim |M_{int} + M_{ext}|^2 \)

Internal part: \( M_{int} = \int_0^a u_j(r, E) u_f(r) r^\lambda dr \)

External part: \( M_{ext} = \int_a^\infty u_j(r, E) u_f(r) r^\lambda dr \)

with \( u_f(r) = C W_{-\eta_B, \ell+1/2}(2k_B r) \)
\( u_j(r, E) = F_j(kr) \cos \delta_j + G_j(kr) \sin \delta_j \)

At low energies: \( \delta_j \approx 0 \rightarrow \) the ANC \( C \) determines the external contribution

Different options

- \( M_{int} \) dominant, \( M_{ext} \) negligible: high energies, resonant reactions
- \( M_{int} \) and \( M_{ext} \) important
- \( M_{int} \) negligible, \( M_{ext} \) dominant: low binding energies
  \( \rightarrow \) « peripheral reaction »
  \( \rightarrow \) amplitude of cross section determined by the ANC
  \( \rightarrow \) energy dependence determined by the Coulomb functions
  \( \rightarrow \) measurement of the ANC provides the cross section at low energies
7. Radiative capture in the potential model

Bound state:
E = -0.137 MeV
wave function \( u_f (r) \)

Initial state \( E_i = 0.1 \) and 2 MeV
wave function \( u_i (r, E) \)

\( ^7\text{Be}(p, \gamma)^8\text{B} \)

Integrant
\( u_i (r, E)u_f (r)r^3 \)
7. Radiative capture in the potential model

Integrand for 3 reactions at low energies

\[ I(\rho) = u_{J_i}(\rho, E)u_{J_f}(\rho)\rho^\lambda \]

\(^7\text{Be}(p,\gamma)^8\text{B}\) external (low binding energy)

- \(^{12}\text{C}(p,\gamma)^{13}\text{N}\) internal: resonant reaction
- \(^3\text{He}(\alpha,\gamma)^7\text{Be}\): both components are important

Qualitatively true for all models (only the short-range part differs)
8. The R-matrix method
• General presentation
• Single resonance system
• Applications to elastic scattering $^{12}\text{C}+p$
• Application to $^{12}\text{C}(p,\gamma)^{13}\text{N}$ and $^{12}\text{C}(\alpha,\gamma)^{16}\text{O}$
8. The R-matrix method

- Introduced by Wigner (1937) to parametrize resonances (nuclear physics)
  In nuclear astrophysics: used to fit data

- Provides scattering properties at all energies (not only at resonances)

- Based on the existence of 2 regions (radius a): 
  - Internal: coulomb+nuclear
  - external: coulomb

\[ \text{Coulomb} \]
\[ \text{Nuclear+Coulomb: R-matrix parameters} \]

\[ \text{Exit channels:} \]
- \( ^{12}\text{C}(2^+)+\alpha \)
- \( ^{12}\text{C}+\alpha \)
- \( ^{15}\text{N}+p, ^{15}\text{O}+n \)

\[ \text{Entrance channel:} \]
- \( ^{12}\text{C}+\alpha \)
8. The R-matrix method

Main Goal: fit of experimental data

$^{18}\text{Ne}+p$ elastic scattering $\rightarrow$ resonance properties

Nuclear astrophysics: $^{12}\text{C}(\alpha,\gamma)^{16}\text{O}$ $\rightarrow$ Extrapolation to low energies
8. The R-matrix method

- **Internal region**: The R matrix is given by a set of resonance parameters $E_i, \gamma_i^2$

  \[ R(E) = \sum_i \frac{\gamma_i^2}{E_i - E} = a \frac{\Psi'(a)}{\Psi(a)} \]

  \[
  \begin{array}{c}
  i=3, E_3, \gamma_3^2 \\
  i=2, E_2, \gamma_2^2 \\
  i=1, E_1, \gamma_1^2 \\
  \end{array}
  \]

- **External region**: Coulomb behaviour of the wave function

  \[ \Psi(r) = I(r) - UO(r) \]

  \[ \rightarrow \text{the collision matrix } U \text{ is deduced from the R-matrix (repeated for each spin/parity } J\pi) \]

- Two types of applications:
  - **Phenomenological R matrix**: $\gamma_i^2$ and $E_i$ are fitted to the data (astrophysics)
  - **Calculable R matrix**: $\gamma_i^2$ and $E_i$ are computed from basis functions (scattering theory)

- R-matrix radius $a$ is not a parameter: the cross sections must be insensitive to $a$

- Can be extended to multichannel calculations (transfer), capture, etc.

- Well adapted to nuclear astrophysics: low energies, low level densities
8. The R-matrix method

A simple case: elastic scattering with a single isolated resonance

- From the total width $\Gamma \rightarrow$ reduced width $\Gamma = 2\gamma^2 P_l(E_R)$
  
  $P_l(E_R) =$ penetration factor

- Link between $(E_R, \gamma^2) \leftrightarrow (E_0, \gamma_0^2)$

- Calculation of the R-matrix $R(E) = \frac{\gamma_0^2}{E_0 - E}$

- Calculation of the scattering matrix: $U(E) = \frac{I(ka)}{O(ka)} \frac{1 - L^* R(E)}{1 - LR(E)}$ (must be done for each $\ell$)

- Calculation of the cross section $\rightarrow E_0$ and/or $\gamma_0^2$ can be fitted
Example: $^{12}\text{C} + \text{p}: E_R = 0.42 \text{ MeV}$

In the considered energy range: resonance $J=1/2^+ (\ell = 0)$
- Phase shift for $\ell = 0$ is treated by the R matrix
- Other phase shifts $\ell > 0$ are given by the hard-sphere approximation
8. The R-matrix method

First example: Elastic scattering $^{12}$C+p

Data from H.O. Meyer et al., Z. Phys. A279 (1976) 41

![Graphs showing scattering cross-sections for different energies and angles.]

R matrix fits for different channel radii

<table>
<thead>
<tr>
<th>$a$</th>
<th>$E_R$</th>
<th>$\Gamma$</th>
<th>$E_0$</th>
<th>$\gamma_0^2$</th>
<th>$\chi^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.5</td>
<td>0.4273</td>
<td>0.0341</td>
<td>-1.108</td>
<td>1.334</td>
<td>2.338</td>
</tr>
<tr>
<td>5</td>
<td>0.4272</td>
<td>0.0340</td>
<td>-0.586</td>
<td>1.068</td>
<td>2.325</td>
</tr>
<tr>
<td>5.5</td>
<td>0.4272</td>
<td>0.0338</td>
<td>-0.279</td>
<td>0.882</td>
<td>2.321</td>
</tr>
<tr>
<td>6</td>
<td>0.4271</td>
<td>0.0336</td>
<td>-0.085</td>
<td>0.745</td>
<td>2.346</td>
</tr>
</tbody>
</table>

$\Rightarrow E_R, \Gamma$ very stable with $a$

$\Rightarrow$ global fit independent of $a$
8. The R-matrix method

Extension to transfer, example: $^{18}\text{F}(p,\alpha)^{15}\text{O}$

- Link between $(E_R, \gamma_1^2, \gamma_2^2) \leftrightarrow E_0, \gamma_{01}^2, \gamma_{02}^2$ more complicated

  
  
  **R-matrix: 2x2 matrix**

  
  
  $R_{ii}(E) = \gamma_{01}^2 \over E_0 - E$ associated with the entrance channel

  $R_{ff}(E) = \gamma_{02}^2 \over E_0 - E$ associated with the exit channel

  $R_{if}(E) = \gamma_{01}\gamma_{02} \over E_0 - E$ associated with the transfer

- Scattering matrix: 2x2: $U_{11, U_{22}} \rightarrow$ elastic cross sections

  $U_{12,} \rightarrow$ transfer cross section

- More parameters, but some are common to elastic scattering $(E_0, \gamma_{01}^2)$

  $\rightarrow$ constraints with elastic scattering
Recent application to $^{18}$F(p,p)$^{18}$F and $^{18}$F(p,α)$^{15}$O

The R-matrix method

Simultaneous fit of both cross sections
angle: 176°
for each resonance: $J\pi, E_R, \Gamma_p, \Gamma_\alpha$
8 resonances $\rightarrow$ 24 parameters
Extension to radiative capture

\[ E \]

resonance: energy \( E_R \), reduced \( \gamma^2 \), gamma width \( \Gamma_\gamma \) = “observed parameters”

“calculated” parameters: \( E_0, \gamma_0^2, \Gamma_{0\gamma} \)

threshold

Final state: ANC \( C_f \)

Capture reaction = transition between an initial state at energy \( E \) to bound states

Cross section \( \sigma_C(E) \sim |< \Psi_f | H_\gamma | \Psi_i(E) >|^2 \)

Additional pole parameter: gamma width \( \Gamma_{\gamma i} \)

\[ < \Psi_f | H_\gamma | \Psi_i(E) > = < \Psi_f | H_\gamma | \Psi_i(E) >_{int} + < \Psi_f | H_\gamma | \Psi_i(E) >_{ext} \]

internal part: \( < \Psi_f | H_\gamma | \Psi_i(E) >_{int} \sim \sum_{i=1}^{N} \frac{\gamma_i \sqrt{\Gamma_{\gamma i}}}{E_i - E} \)

external part: \( < \Psi_f | H_\gamma | \Psi_i(E) >_{ext} \sim C_f \int_{a}^{\infty} W (2k_f r) r^\lambda (I_i(kr) - UO_i(kr)) dr \)
8. The R-matrix method

External part: $< \Psi_f | H_\gamma | \Psi_i (E) >_{ext} \sim C_f \int_a^\infty W(2k_f r) r^\lambda (I_i(kr) - UO_i(kr)) dr$

Essentially depends on $k_f$

\[ E_f = \frac{\hbar^2}{2\mu} k_f^2 \]

Witthaker function $W(2k_f r) \sim \exp(-k_f r)$

- $k_f$ large: fast decrease
  
  example $^{12}\text{C}(\alpha,\gamma)^{16}\text{O}$, $E_f = 7.16 \text{ MeV}$, $\mu = 3$  → external term negligible
  
  → insensitive to $C_f$

- $k_f$ small: slow decrease
  
  example: $^7\text{Be}(p,\gamma)^8\text{B}$, $E_f = 0.137 \text{ MeV}$, $\mu = 7/8$  → external term dominant
  
  → mainly given by $C_f$

- Contribution of internal/external terms depends on energy (external larger at low energies)
8. The R-matrix method

Example 1: $^{12}\text{C}(p,\gamma)^{13}\text{N}$: R-matrix calculation with a single pole

Experiment: $E_R = 0.42 \text{ MeV}$, $\Gamma_p = 31 \text{ keV}$, $\Gamma_\gamma = 0.4 \text{ eV}$

Red line: internal contribution, pure Breit-Wigner approximation
Green lines: external contribution: important at low energies, sensitive to the ANC
8. The R-matrix method

Example 2: $^{12}\text{C}(\alpha,\gamma)^{16}\text{O}$

General presentation of $^{12}\text{C}(\alpha,\gamma)^{16}\text{O}$

- Determines the $^{12}\text{C}/^{16}\text{O}$ ratio
- Cross section needed near $E_{\text{cm}}=300$ keV (barrier $\sim 2.5$ MeV) → cannot be measured in the Gamow peak
- $1^-$ and $2^+$ subthreshold states
  → extrapolation difficult
- $E1$ and $E2$ important (E1 forbidden when $T=0$)
- Interferences between $1^-_1, 1^-_2$ and between $2^+_1, 2^+_2$
- Capture to gs dominant but also cascade transitions

$^{12}\text{C}+\alpha$
8. The R-matrix method

Many experiments

- **Direct** $^{12}\text{C}(\alpha,\gamma)^{16}\text{O}$ (angular distributions are necessary: E1 and E2)

- **Indirect**: spectroscopy of $1^-_1$ and $2^+_1$ subthreshold states

- **Constraints**
  - $\alpha+^{12}\text{C}$ phase shifts ($1^- \rightarrow E1$, $2^+ \rightarrow E2$)
  - E1: $^{16}\text{N}$ beta decay
    probes $J=1^-$ $\rightarrow$ E1

- E2: ???
8. The R-matrix method

Current situation

![Graph showing the current situation with data points for different E1 and E2 channels. The graph plots S (keV b) against E_{cm} (MeV) for various authors and parameters.](image)

The R-matrix method
8. The R-matrix method

S(300 keV): current situation for E1

$^{12}\text{C}(\alpha,\gamma)^{16}\text{O}$ E1

Weisser 74
Dyer 74
Koonin 74
Weisser 74
Dyer 74
Koonin 74
Humblet 76
Descouvement 87
Redder 87
Kremer 88
Filippone 89
Ouellet 92
Humblet 93
Azuma 94
Ouellet 96
Trautvetter 97
Gialanella 01
Fey 03

$^{16}\text{N}$ data available
8. The R-matrix method

$S(300 \text{ keV})$: current situation for E2

![Graph showing $S_{E2}$ (keV-barn) versus year for different research publications.](image)
9. Microscopic models
9. Microscopic models

- Goal: solution of the Schrödinger equation $H\Psi = E\Psi$
- Hamiltonian: $H = \sum_i T_i + \sum_{j>i} V_{ij}$
  - $T_i$ = kinetic energy of nucleon $i$
  - $V_{ij}$ = nucleon-nucleon interaction

- **Cluster** approximation $\Psi = A\phi_1\phi_2g(\rho)$
  - $\phi_1, \phi_2$ = internal wave functions (**input, shell-model**)
  - $g(\rho)$ = relative wave function (**output**)
  - $A$ = antisymmetrization operator

- Generator Coordinate Method (GCM): the radial function is expanded in Gaussians
  → Slater determinants (well adapted to numerical calculations)
- Microscopic R-matrix: extension of the standard R-matrix → reactions
9. Microscopic models

Many applications: not only nuclear astrophysics spectroscopy, exotic nuclei, elastic and inelastic scattering, etc.

Extensions:
• Multicluster calculations: $\rightarrow$ deformed nuclei (example: $^7\text{Be}+\text{p}$)

\[ \Psi = A \phi_1 \phi_2 g(\rho) + A \phi_1^* \phi_2^* g^*(\rho) + \cdots \]
$\rightarrow$ better wave functions
$\rightarrow$ inelastic scattering, transfer

• Multichannel calculations:

• Ab initio calculations: no cluster approximation
  $\rightarrow$ very large computer times
  $\rightarrow$ limited to light nuclei
  $\rightarrow$ difficult for scattering (essentially limited to nucleon-nucleus)
9. Microscopic models

Example: $^7\text{Be}(p,\gamma)^8\text{B}$

- Important for the solar-neutrino problem
- Since 1995, many experiments:
  - Direct (proton beam on a $^7\text{Be}$ target)
  - Indirect (Coulomb break-up)
- Extrapolation to zero energy needs a theoretical model (energy dependence)

From E. Adelberger et al., Rev. Mod. Phys. 83 (2011) 196
Example: $^7\text{Be}(p,\gamma)^8\text{B}$

- Microscopic cluster calculations: 3-cluster calculations

- Includes the deformation of $^7\text{Be}$: cluster structure $\alpha + ^3\text{He}$
- Includes rearrangement channels $^5\text{Li} + ^3\text{He}$
- Can be applied to $^8\text{B}/^8\text{Li}$ spectroscopy
- Can be applied to $^7\text{Be}(p,\gamma)^8\text{B}$ and $^7\text{Li}(n,\gamma)^8\text{Li}$

$^7\text{Be} = \alpha + ^3\text{He}$

$^5\text{Li} = \alpha + p$
9. Microscopic models

Spectroscopy of $^8\text{B}$

<table>
<thead>
<tr>
<th></th>
<th>experiment</th>
<th>Volkov</th>
<th>Minnesota</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu$ ($^2_+^\pm$) ($\mu_N$)</td>
<td>1.03</td>
<td>1.48</td>
<td>1.52</td>
</tr>
<tr>
<td>$Q(2^+_\pm)$ (e.fm$^2$)</td>
<td>6.83 ± 0.21</td>
<td>6.6</td>
<td>6.0</td>
</tr>
<tr>
<td>$B(M1,1^+<em>\pm\rightarrow2^+</em>\pm)$ (W.u.)</td>
<td>5.1 ± 2.5</td>
<td>3.4</td>
<td>3.8</td>
</tr>
</tbody>
</table>

Channel components in the $^8\text{B}$ ground state

<p>| | |</p>
<table>
<thead>
<tr>
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<th></th>
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</thead>
<tbody>
<tr>
<td>$^7\text{Be}(3/2^-)+p$</td>
<td>47%</td>
</tr>
<tr>
<td>$^7\text{Be}(1/2^-)+p$</td>
<td>9%</td>
</tr>
<tr>
<td>$^5\text{Li}(3/2^-)+^3\text{He}$</td>
<td>34%</td>
</tr>
<tr>
<td>$^5\text{Li}(1/2^-)+^3\text{He}$</td>
<td>3%</td>
</tr>
</tbody>
</table>

⇒ Important role of the 5+3 configuration
9. Microscopic models

\[ ^7\text{Be}(p,\gamma)^8\text{B} \] S factor

- Low energies (E<100 keV): energy dependence given by the Coulomb functions
- 2 NN interactions (MN, V2): \( \rightarrow \) the sensitivity can be evaluated
- Overestimation: due to the \(^8\text{B}\) ground state (cluster approximation)
9. Microscopic models

Cluster models
• In general a good approximation, but do not allow the use of realistic NN interactions
• Example: $\alpha$ particle described by 4 0s orbitals
  → intrinsic spin =0
  → no spin-orbit, no tensor force, no 3-body force
  → these terms are simulated by (central) NN interactions

Ab initio models
• No cluster approximation
• Use of realistic NN interactions (fitted on deuteron, NN phase shifts, etc.)
• Application: d+d systems $^2\text{H}(d,\gamma)^4\text{He}$, $^2\text{H}(d,p)^3\text{H}$, $^2\text{H}(d,n)^3\text{He}$
  two physics issues
  • Analysis of the d+d S factors (Big-Bang nucleosynthesis)
  • Role of the tensor force in $^2\text{H}(d,\gamma)^4\text{He}$
$^2\text{H}(d,\gamma)^4\text{He}$ S factor

- Ground state of $^4\text{He}=0^+$
- E1 forbidden $\rightarrow$ main multipole is $E2 \rightarrow 2^+ \rightarrow 0^+$ transition $\rightarrow$ d wave as initial state
- Experiment shows a plateau below 0.1 MeV: typical of an s wave
- Interpretation: the $^4\text{He}$ ground state contains an admixture of d wave

Initial $2^+$ state: $\Psi^{2^+} = \Psi^{2^+}(L = 2, S = 0) + \Psi^{2^+}(L = 0, S = 2) = |2^+, 0 > + |2^+, 2 >$

Final $0^+$ state: $\Psi^{0^+} = \Psi^{0^+}(L = 0, S = 0) + \Psi^{0^+}(L = 2, S = 2) = |0^+, 0 > + |0^+, 2 >$

$E2$ matrix element $< \Psi^{0^+} | E2 | \Psi^{2^+} >$

$\approx < 0^+, 0 | E2 | 2^+, 0 >: d \rightarrow s$, dominant $E > 100$ keV

$+ < 0^+, 2 | E2 | 2^+, 0 >: s \rightarrow d$, tensor ($E < 100$ keV)

$\rightarrow$ direct effect of the tensor force
9. Microscopic models

Application: d+d systems
  *Phys. Rev. Lett. 107 (2011) 132502*
• Mixing of d+d, $^3$H+p, $^3$He+n configurations

- The total wave function is written as an expansion over a gaussian basis
- Superposition of several angular momenta
- 4-body problem (in the cluster approximation we would have: $x_1=x_2=0$)
We use 3 NN interactions:

• Realistic: Argonne AV8’, G3RS
• Effective: Minnesota MN

- No parameter
- MN does not reproduce the plateau (no tensor force)
- D wave component in $^4$He:
  13.8% (AV8’)
  11.2% (G3RS)
9. Microscopic models

Transfer reactions $^2\text{H}(d,p)^3\text{H}$, $^2\text{H}(d,n)^3\text{He}$
Needs for nuclear astrophysics:

- low energy cross sections
- resonance parameters

Experiment: direct and indirect approaches

Theory: various techniques

- fitting procedures (R matrix) → extrapolation
- non-microscopic models: potential, DWBA, etc.
- microscopic models:
  - cluster: developed since 1960’s, applied to NA since 1980’s
  - ab initio: problems with scattering states, resonances → limited at the moment
- Current challenges: new data on \(^3\text{He}(\alpha,\gamma)^7\text{Be}, \text{ triple } \alpha \text{ process, } ^{12}\text{C}(\alpha,\gamma)^{16}\text{O}, \text{ etc.} \)
  \(D(d,\gamma)^4\text{He} : 4 \text{ nucleons } \rightarrow 4 \text{ clusters}\)