Relativistic Quantum Mechanics (Slides only)

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Contents

I. FERMIONS

A. Heuristic derivation of Dirac equation

$$i\partial_0\psi = H\psi, \quad H = \boldsymbol{\alpha p} + \beta m \tag{1}$$

"Square root" of the Klein-Gordon equation $\partial_0^2\phi = \Delta\phi - m^2\phi$:

$$-\partial_0^2\psi = (-i\boldsymbol{\alpha}\cdot\boldsymbol{\partial} + \beta m)^2\psi \tag{2}$$

where $\{A, B\} = AB + BA$ and

$$\begin{aligned} \alpha_j\alpha_k\partial_j\partial_k &= \frac{1}{2}(\{\alpha_j,\alpha_k\} + [\alpha_j,\alpha_k])\frac{1}{2}(\{\partial_j,\partial_k\} + [\partial_j,\partial_k]) \\ &= \frac{1}{4}(\{\alpha_j,\alpha_k\} + [\alpha_j,\alpha_k])\{\partial_j,\partial_k\} \\ &= \frac{1}{2}\{\alpha_j,\alpha_k\}\partial_j\partial_k \end{aligned} \tag{3}$$

$$\begin{aligned} -\partial_0^2\psi &= [-\{\alpha_j,\alpha_k\}\partial_j\partial_k + \beta^2m^2 - mi\{\alpha_j,\beta\} + \partial_j]\psi = -\Delta\phi + m^2\phi \\ \Longrightarrow &\quad \{\alpha_j,\alpha_k\} = 2\delta_{j,k}, \quad \beta^2 = \mathbb{1}, \quad \{\boldsymbol{\alpha},\beta\} = 0. \end{aligned} \tag{4}$$

Covariant notation: $\gamma^\mu = (\beta, \beta\boldsymbol{\alpha})$,

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = (i\gamma^0\partial_0 + i\boldsymbol{\gamma\nabla} - m)\psi(x) = (i\partial\!\!\!/ - m)\psi(x) = 0 \tag{5}$$

where

$$\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu} = 2\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{6}$$

transforms as contravariant four-vector.

N.B.

$$(i\partial\!\!\!/ - m)(i\partial\!\!\!/ + m) = -\partial^2 - m^2. \tag{7}$$

The simplest realization is in terms of 4×4 matrices:

$$\beta = \gamma^0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \qquad \boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \qquad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}, \tag{8}$$

Hermitian conjugate:

$$i\partial_\mu \psi^\dagger(x)\gamma^{\mu\dagger} + \psi(x)m = 0 \tag{9}$$

non covariant equation because $\gamma^{0\dagger} = \gamma^0$, $\gamma^{j\dagger} = -\gamma^j$

Dirc conjugate: $\bar{\psi} = \psi^\dagger\gamma^0$, $\gamma^0\gamma^\mu\gamma^0 = \gamma^{\mu\dagger}$,

$$i\partial_\mu \bar{\psi}\gamma^\mu + \bar{\psi}m = 0 \tag{10}$$

covariant equation

Modified conjugate:

- Klein-Gordon: Hermitian Hamiltonian,
- Dirac: covariant E.O.M.

$$\langle\psi|\psi'\rangle = \int d^3x \psi^\dagger(\boldsymbol{x})\psi'(\boldsymbol{x}). \tag{11}$$

Lagrangian:

$$L = \frac{i}{2}[\bar{\psi}\gamma^\mu(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma^\mu\psi] - m\bar{\psi}\psi. \tag{12}$$

Noether current of the $U(1)$ symmetry, $\psi(x) \to e^{i\theta}\psi(x)$, $\bar{\psi}(x) \to e^{-i\theta}\bar{\psi}(x)$:

$$j^\mu = \bar{\psi}\gamma^\mu\psi \tag{13}$$

N.B. $j^0 = \psi^\dagger\psi \geq 0$

B. Non-relativistic spinors

- **Elementary systems:** Irreducible representations of the external symmetries
  1. Representation: $U(g) : G \to \mathcal{H}$, $U(gg') = U(g)U(g')$
  2. Irreducible (star) representation: $\exists\ |\psi_0\rangle \in \mathcal{H}$, $\forall\ |\psi\rangle \in \mathcal{H}$, $\exists\ g_\psi \in G$, $|\psi\rangle = U(g_\psi)|\psi_0\rangle$, $g \in G$
  3. Symmetry group: Connected subgroup and disconnected cosets

     first find the representation first the connected subgroup

     next for the cosets

- $SU(2)$:

  1. Definition:

$$A(a,\boldsymbol{a}) = a\mathbb{1} + i\boldsymbol{a\sigma} = \begin{pmatrix} a+ia_3 & ia_1+a_2 \\ ia_1-a_2 & a-ia_3 \end{pmatrix}, \quad a,\boldsymbol{a}\in R, \quad \det A(a,\boldsymbol{a}) = a^2+\boldsymbol{a}^2 = 1 \tag{14}$$

  2. $\dim(SU(2)) = 3$

  3. Multiplication: $\sigma_a\sigma_b = \mathbb{1}\delta_{ab} + i\epsilon_{abc}\sigma_c$

$$\begin{aligned} A(a,\boldsymbol{a})A(b,\boldsymbol{b}) &= (a\mathbb{1}+i\boldsymbol{a\sigma})(b\mathbb{1}+i\boldsymbol{b\sigma}) \\ &= (ab - \boldsymbol{ab} + i(a\boldsymbol{b} + b\boldsymbol{a} - \boldsymbol{a}\times\boldsymbol{b})\boldsymbol{\sigma} \\ &= A(ab-\boldsymbol{ab}, a\boldsymbol{b}+b\boldsymbol{a}-\boldsymbol{a}\times\boldsymbol{b}). \end{aligned} \tag{15}$$

  4. Inverse: $A^{-1}(a,\boldsymbol{a}) = A(a,-\boldsymbol{a}) = A^\dagger(a,\boldsymbol{a}) \Longrightarrow \{A\} = SU(2)$

  5. Another parametrization: spin $s=\frac{1}{2}$, $\boldsymbol{n}^2 = 1$,

$$A_{\boldsymbol{n}}(\alpha) = e^{-\frac{i}{2}\alpha\boldsymbol{n\sigma}} = \mathbb{1}\cos\frac{\alpha}{2} - i\boldsymbol{n\sigma}\sin\frac{\alpha}{2} = A\left(\cos\frac{\alpha}{2} - \boldsymbol{n}\sin\frac{\alpha}{2}\right) \tag{16}$$

- **Fundamental representation:** $\dim(SU(2)) = 3$ $\psi \to A\psi$,

- **Complex conjugate fundamental representation:** $\eta \to A^*\eta$, unitary equivalent of the fundamental representation, $U'(g) = V^\dagger U(g)V$

$$(i\boldsymbol{\sigma})^* = \sigma_2 i\boldsymbol{\sigma}\sigma_2 = \sigma_2^\dagger i\boldsymbol{\sigma}\sigma_2 \quad \Longrightarrow \quad A^* = \sigma_2 A\sigma_2 \tag{17}$$

- **Adjoint representation:** X_{ab}, $a,b = 1,2$,

$$X \to AXA^\dagger, \tag{18}$$

  8 dimensional reducible representation, but if $X^\dagger = X \Longrightarrow$is $AXA^\dagger = (AXA^\dagger)^\dagger$

  4 dimensional representation by Hermitean matrices:

$$X(x^\mu) = x^0\mathbb{1} + \boldsymbol{x\sigma} = \begin{pmatrix} x^0+x^3 & x^1-ix^2 \\ x^1+ix^2 & x^0-x^3 \end{pmatrix}, \tag{19}$$

  with real coefficients x^μ, a spin wave function of two spin half particles, $\frac{1}{2}\otimes\frac{1}{2} = 0\oplus 1 = a\oplus\boldsymbol{a}$

- **Relation between $SO(3)$ and $SU(2)$, Part I.:** The triplet multiplet is a three-vector

1. length $\det X = -\boldsymbol{x}^2$ is preserved: $\det X \to \det AXA^\dagger = \det A \det X \det A^\dagger = \det X$

2. infinitesimal $SU(2)$ transformations are infinitesimal $SO(3)$ rotations:

$$\begin{aligned} A_{\boldsymbol{n}}(\alpha)X(0,\boldsymbol{x})A^\dagger_{\boldsymbol{n}}(\alpha) &\approx \left(\mathbb{1} - i\frac{\alpha}{2}\boldsymbol{n}\boldsymbol{\sigma}\right)\boldsymbol{x}\boldsymbol{\sigma}\left(\mathbb{1} + i\frac{\alpha}{2}\boldsymbol{n}\boldsymbol{\sigma}\right) \\ &\approx \boldsymbol{x}\boldsymbol{\sigma} - i\frac{\alpha}{2}[\boldsymbol{n}\boldsymbol{\sigma}, \boldsymbol{x}\boldsymbol{\sigma}] \\ &= X(0, \boldsymbol{x} + \alpha\boldsymbol{n}\times\boldsymbol{x}), \end{aligned} \tag{20}$$

The repeated application establishes the equivalence for finite α.

- **Fundamental group:** Group over the homotopy classes of closed paths: Γ is a topological space (each point is contained in a neighborhood, i.e. an open set)

1. $\gamma : [0,1] \to \Gamma$, $\gamma(0) = \gamma(1)$

2. γ_1 homotopic with γ_2, $\gamma_1 \approx \gamma_2$, if

   (a) γ_1 can continuously be deformed to γ_2 within Γ, or

   (b) $\exists\, f : [0,1]\otimes[0,1] \to \Gamma$ continuous function such that

$$f(s,0) = \gamma_1(s), \quad f(s,1) = \gamma_2(s) \tag{21}$$

3. Equivalence classes of homotopically equivalent loops $G_\gamma = \{\gamma' | \gamma' \approx \gamma\}$.

4. Multiplication of loops:

$$\gamma_2 \otimes \gamma_1(s) = \begin{cases} \gamma_1(2s) & 0 < s < \frac{1}{2} \\ \gamma_2(2s-1) & \frac{1}{2} < s < 1 \end{cases} \tag{22}$$

   e.g. paths on the circle with a given winding number

5. $\gamma_2 \otimes \gamma_1 = \gamma_3 \quad \leftrightarrow \quad G_{\gamma_2} \otimes G_{\gamma_1} = G_{\gamma_3}$ is consistent.

6. Fundamental group, $\pi_1(\Gamma)$: Homotopy classes with the multiplication and $\gamma^{-1}(s) = \gamma(1-s)$. e.g. $H_1(U(1)) = Z$.

- **Relation between $SO(3)$ and $SU(2)$, Part II.:**

1. The mapping $A \to R(A)$, $SU(2) \to SO(3) \subset O(3)$ is a two-to-one, $R(-A) = R(A)$.

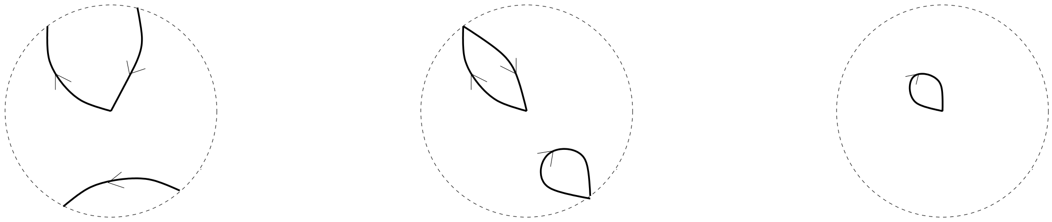
2. The group manifold $SU(2)$ is simply connected, $\pi_1(SU(2)) = \mathbb{1}$.

3. The group manifold $O(3)$ is doubly connected, $\pi_1(O(3)) = Z_2 = \{1, -1\}$.

   Proof:

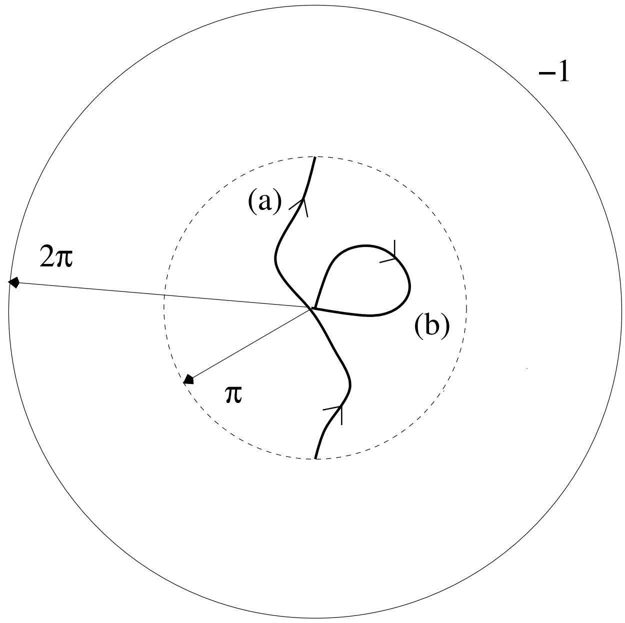
(a) $R_{\boldsymbol{n}}(\pi), R_{\boldsymbol{n}}(-\pi) \in SO(3)$, are identical, $R_{\boldsymbol{n}}(\pi) = R_{\boldsymbol{n}}(-\pi)$ because a rotation by 2π leaves a vector unchanged.

(b) Continuous deformation of the loop can change n_d, the number of diametrically opposite points of $SO(3)$ visited in units of 2.



(c) The parity n_d mod 2 is topological invariant.

4. $SO(3):\ |\alpha| < \pi$, $SU(2):\ |\alpha| < 2\pi$



C. Multi-valued wave functions

Wave function: $\psi(\boldsymbol{x}) : R^3 \to C$ possible Riemann-sheets, e.g. $\ln z$. Do they play role in physics?

Particle on the circle: $\boldsymbol{x} = r(\cos\phi, \sin\phi)$, $\psi(\phi) = \langle\phi|\psi\rangle$

- Hermitean extension defines different Hilbert spaces: $p = \frac{\hbar}{ir}\partial_\phi = p^\dagger$,

$$\langle\psi_1|p|\psi_2\rangle = \int_{-\pi}^{\pi} d\phi \psi_1^*(\phi)\frac{\hbar}{ir}\partial_\phi\psi_2(\phi)$$

$$
\begin{aligned}
&= \int_{-\pi}^{\pi} d\phi \left(\frac{\hbar}{ir}\partial_\phi \psi_1(\phi)\right)^* \psi_2(\phi) + \frac{\hbar}{ir}\psi_1^*(\phi)\psi_2(\phi)\Big|_{-\pi}^{\pi} \\
&= \langle \psi_1 | p^\dagger | \psi_2 \rangle \\
\Longrightarrow \mathcal{H}_\theta &= \{\psi(\phi) | \psi(\phi + 2\pi) = e^{i\theta}\psi(\phi)\} \qquad (23)
\end{aligned}
$$

- $\mathcal{H}_\theta$: $\psi_n(\phi) = e^{i(n+\frac{\theta}{2\pi})\phi} \in \mathcal{H}_\theta$, When the momentum operator and the free Hamiltonian, $H = \frac{p^2}{2m}$, are restricted into then its eigenstates are with the eigenvalues

$$
\begin{aligned}
p_\theta \psi_n &= \frac{\hbar}{r}\left(n + \frac{\theta}{2\pi}\right)\psi_v \\
H_\theta \psi_n &= \frac{\hbar^2}{2mr^2}\left(n + \frac{\theta}{2\pi}\right)^2 \psi_n \qquad (24)
\end{aligned}
$$

  The θ-dependence is periodic, $\psi_n(\phi) \to \psi_{n+1}(\phi)$ as $\theta \to \theta + 2\pi$.

- Physical origin: The particle interferes with itself as it turns around the circle. No classical analogy.

- Particle interferes with itself: What matters in classical physics is where you are. However in quantum physics it is important how did you get there, as well.

- Quantum symmetry: $\phi \to \phi + 2\pi$, irreducible representations: $\psi(\phi + 2\pi) = e^{i\theta}\psi(\phi)$ (like translations)

Aharonov-Bohm effect:

- Particle on a ring which encircles a magnetic flux

$$
\Phi = \int_\Sigma d\boldsymbol{n}\boldsymbol{B}(\boldsymbol{x}) = \oint_\mathcal{R} d\boldsymbol{x}\boldsymbol{A}(\boldsymbol{x}) \qquad (25)
$$

- The magnetic field is vanishing *on the circle*, $A = A_\phi = \frac{\Phi}{2\pi r} = \partial_{r\phi}\alpha(\phi)$ $\alpha(\phi) = \frac{\phi\Phi}{2\pi}$ (pure gauge).

- Hamiltonian, $\boldsymbol{p} \to \boldsymbol{p} - \frac{e}{c}\boldsymbol{A}$

$$
H_\Phi = \frac{\hbar^2}{2mr^2}\left(\frac{1}{i}\partial_\phi - \frac{e\Phi}{2\pi\hbar c}\right)^2. \qquad (26)
$$

- Aharonov-Bohm effect: The expectation values dependend on the magnetic field.

  1. The magnetic field is vanishing along the path of the propagation.

2. The physical effects of the magnetic flux can be eliminated by an appropriate gauge transformation. These gauge transformation are aperiodic and lead out of the Hilbert space.

- H_Φ is the free Hamiltonian (no magnetic flux) in $\mathcal{H}_{\frac{e\Phi}{\hbar c}}$.
- Relation to quantum mechanics on the circle: $\Phi \quad \Leftrightarrow \quad \theta$

  Gauge transformation:

$$\boldsymbol{A}(\boldsymbol{x}) \to \boldsymbol{A}(\boldsymbol{x}) + \frac{\hbar c}{e}\boldsymbol{\nabla}\chi(\boldsymbol{x}), \quad \psi(\boldsymbol{x}) \to e^{i\chi(\boldsymbol{x})}\psi(\boldsymbol{x}), \tag{27}$$

  - $A_\phi = 0$ and $\chi = \phi\frac{e}{\hbar c}\frac{\Phi}{2\pi} \to A_\phi = \frac{\Phi}{2\pi r}$, magnetic flux Φ.
  - $\mathcal{H}_\theta \to \mathcal{H}_{\theta+\frac{e}{\hbar c}\Phi}$, $H_\theta \to H_{\theta+\frac{e}{\hbar c}\Phi}$.

Dynamical condition of the multi-valuedness of wave function: The particle has to be excluded from a region which makes the coordinate space multiply connected.

Topological symmetry: The fundamental group is a (classical) symmetry and its irreducible representations generate a new quantum number, Θ, e.g. rotor dynamics,

$$\pi_1(SO(3)) = Z_2, \quad \Theta = \begin{cases} 0 & \text{boson}, \\ \pi & \text{fermion}. \end{cases} \tag{28}$$

D. Projective representations

1. The vectors $|\psi\rangle$ and $e^{i\alpha}|\psi\rangle$ represent the same physical state.
2. The representations of symmetries are to be generalized to projective representations,

$$\begin{aligned} U(gg') &= U(g)U(g')e^{i\alpha(g,g')} \\ U(g_3g_2g_1) &= U(g_3)U(g_2g_1) = U(g_3g_2)U(g_1) \\ \alpha(g_3, g_2g_1) + \alpha(g_2, g_1) &= \alpha(g_3, g_2) + \alpha(g_3g_2, g_1). \end{aligned} \tag{29}$$

Question: Can the phase factor be eliminated by a "gauge transformation" in the group space?

$$U(g) \to U(g)e^{i\beta(g)} \tag{30}$$

Necessary conditions:

1. Local (in the vicinity of the identity):

(a) The group has no central charge. The central charge is a term, proportional to the identity in the commutator of the generators,

$$[\tau, \tau'] = c\mathbb{1} + \cdots \tag{31}$$

(b) The central charge of a semi-simple group (has no generators which commute with all the other generators, e.g. $SU(n)$) can be eliminated by the appropriate redefinition of the generators.

2. Global: The topology of the group must be simply connected. In case of a multiply connected topology the phase $\alpha(g, g')$ gives a representation of the fundamental group.

Rotations:

1. $SO(3)$ is semi-simple but doubly connected, $\pi_1(SO(3)) = Z_2 \Longrightarrow \exists$ projective representations.

2. The spin is pseudo-scalar and is left unchanged by space inversion $\Longrightarrow U(P) = z_P \mathbb{1}$.

3. Projective representation: $U^2(P) = \mathbb{1} \in Z_2 \Longrightarrow U^2(P) = \pm\mathbb{1}$, $z_P \in \{1, -1, i, -i\}$.

4. $U(R_{\boldsymbol{n}}(2\pi)) = \mathbb{1}e^{i\Theta}$, $e^{i\Theta} \in \pi_1(SO(3)) = Z_2$, bosons ($\Theta = 0$) or fermions ($\Theta = \pi$) in $d = 3$. $\pi_1(SO(2)) = Z$, $-\pi < \Theta < \pi$, anyons in $d = 2$.

E. Relativistic spinors

Spinor representation of $SL(2, C)$: $\dim(Sl(2, C)) = 6$

- Parametrization:

$$A(a, \boldsymbol{a}) = a\mathbb{1} + i\boldsymbol{a}\boldsymbol{\sigma} = \begin{pmatrix} a + ia_3 & ia_1 + a_2 \\ ia_1 - a_2 & a - ia_3 \end{pmatrix}, \quad a, \boldsymbol{a} \in C, \quad \det A(a, \boldsymbol{a}) = a^2 + \boldsymbol{a}^2 = 1 \tag{32}$$

- Fundamental representation: $\xi \to A\xi$.

- Complex conjugate fundamental representation: $\eta \to A^*\eta$.

- van der Waerden conventions: ξ_a, $\eta^{\dot{a}}$, $a, \dot{a} \in \{1, 2\}$,

$$\begin{aligned}
\xi_a &\to A_a^{\ b}\xi_b, \quad \eta_{\dot{a}} \to A_{\dot{a}}^{*\dot{b}}\psi_{\dot{b}} \\
g_{ab} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_2 = g_{\dot{a}\dot{b}} = -g^{ab} = -g^{\dot{a}\dot{b}} \\
\xi_a\chi^a &= \xi_a g^{ab}\chi_b = \xi_1\chi_2 - \xi_2\chi_1 \to (\xi_a\chi_b - \xi_b\chi_a)A_1^{\ a}A_2^{\ b} = (\xi_1\chi_2 - \xi_1\chi_2)\det A_a^{\ b} = \xi_a\chi^a
\end{aligned} \tag{33}$$

- Adjoint representation:

$$
\begin{aligned}
X &\rightarrow AXA^\dagger \\
X^{a\dot{a}}(x^\mu) &= x^0 \mathbb{1} + \boldsymbol{x\sigma} = \begin{pmatrix} x^0+x^3 & x^1-ix^2 \\ x^1+ix^2 & x^0-x^3 \end{pmatrix} = X^{*\dot{a}a}(x^\mu) \\
X_{a\dot{a}} &= g_{ab}g_{\dot{a}\dot{b}}X^{b\dot{b}}, \qquad g = i\sigma_2, \quad \boldsymbol{\sigma}^{\rm tr} = \boldsymbol{\sigma}^* = -\sigma_2\boldsymbol{\sigma}\sigma_2 = \sigma_2\boldsymbol{\sigma}\sigma_2^{\rm tr} \\
&= i\sigma_2(x^0\mathbb{1} + \boldsymbol{x\sigma})i\sigma_2^{\rm tr} = x^0\mathbb{1} - \boldsymbol{x\sigma}^{\rm tr}.
\end{aligned} \tag{34}
$$

Lorentz group and $SL(2,C)$: $Dim(X) = 4$, X is a four vector x^μ.

1. length $\det X = x^2$ is preserved.

2. infinitesimal $SL(2,C)$ transformations are infinitesimal Lorentz transformation:

   Rotations: $R_{\boldsymbol{n}}(\alpha) = A_{\boldsymbol{n}}(\alpha) = e^{-\frac{i}{2}\alpha\boldsymbol{n\sigma}}$, $\alpha \sim 0$

$$
X(0,\boldsymbol{x}) \rightarrow A_{\boldsymbol{n}}(\alpha)X(0,\boldsymbol{x})A^\dagger_{\boldsymbol{n}}(\alpha) = X(0,\boldsymbol{x} + \alpha\boldsymbol{n}\times\boldsymbol{x}) \tag{35}
$$

   Lorentz boost $\boldsymbol{v} = c\beta\boldsymbol{n}$, $\beta \sim 0$

$$
\begin{aligned}
A_{\boldsymbol{n}}(i\beta) &= e^{\frac{\beta}{2}\boldsymbol{n\sigma}} = \mathbb{1}\cosh\frac{\beta}{2} + \boldsymbol{n\sigma}\sinh\frac{\beta}{2} \\
A_{\boldsymbol{n}}(i\beta)X(x^0,\boldsymbol{x})A^\dagger_{\boldsymbol{n}}(i\beta) &\approx \left(\mathbb{1} + \frac{\beta}{2}\boldsymbol{n\sigma}\right)(x^0\mathbb{1} + \boldsymbol{x\sigma})\left(\mathbb{1} + \frac{\beta}{2}\boldsymbol{n\sigma}\right) \\
&\approx x^0\mathbb{1} + (\boldsymbol{x} + x^0\beta\boldsymbol{n})\boldsymbol{\sigma} + \frac{\beta}{2}\{\boldsymbol{n\sigma}, \boldsymbol{x\sigma}\} \\
&= X(x^0 + \beta\boldsymbol{nx}, \boldsymbol{x} + x^0\beta\boldsymbol{n})
\end{aligned} \tag{36}
$$

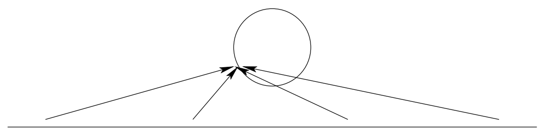
3. Mapping: $SL(2,c) \rightarrow L_+^\uparrow$: two-to-one ($A$ and $-A$).

4. Topology: $L_+^\uparrow = SO(3) \otimes R^3$, $\pi_1(L_+^\uparrow) = \pi_1(SO(3)\otimes R^3) = Z_2$.

Universal covering space:

1. Γ and Γ_c are locally identical, Γ_c is globally simply connected, $\pi_1(\Gamma_c) = \mathbb{1}$

2. $R \rightarrow U(1)$ locally identical, $\pi_1(R) = \mathbb{1}$, $\pi_1(U(1)) = Z$.



3. $SU(2) \rightarrow SO(3)$ locally identical, $\pi_1(SU(2)) = \mathbb{1}$, $\pi_1(SO(3)) = Z_2$.

4. $SL(2,C) \to L_+^\uparrow$ locally identical, $\pi_1(SL(2,C)) = \mathbb{1}$, $\pi_1(L_+^\uparrow) = \pi_1(SO(3) \otimes R^3) = Z_2$.

Representations of the Lorentz group: $L = L_+^\uparrow \oplus L_+^\downarrow \oplus L_-^\uparrow \oplus L_-^\downarrow$

1. Representation of $L_+^\uparrow$:

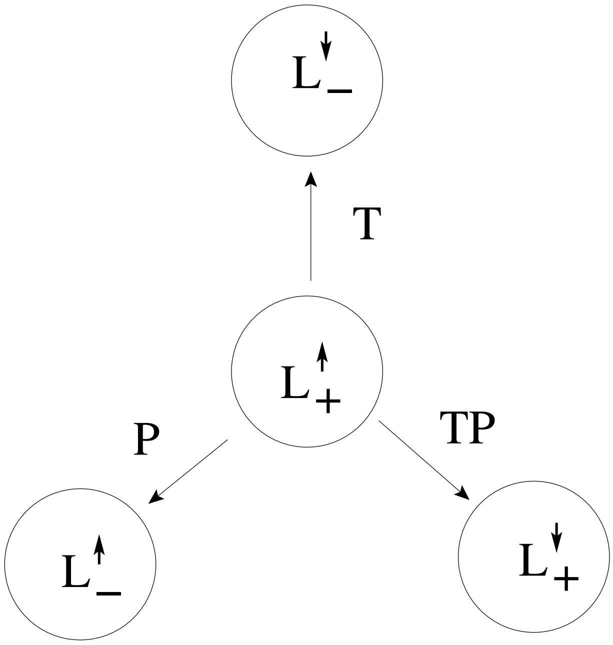
   (a) Simply connected $\pi_1(SL(2,C) \otimes R^3) = \mathbb{1}$

   (b) Doubly connected $\pi_1(SO(3) \otimes R^3) = \pi_1(L_+^\uparrow) = Z_2$

   $\Longrightarrow$projective representations (bosons, fermions)

2. Representation of discrete inversions P, T

3. Representation of the cosets $L_+^\downarrow = TPL_+^\uparrow$, $L_-^\uparrow = PL_+^\uparrow$, $L_-^\downarrow = TL_+^\uparrow$



Representation of discrete symmetries by bi-spinors:

1. P:

   (a) Nonrelativistic case: $P = z_P \mathbb{1}$

   (b) Relativistic case: $PL(\boldsymbol{v}) = L(-\boldsymbol{v})P \neq L(\boldsymbol{v})P \Longrightarrow P \neq z_P \mathbb{1}$

   (c) Irreducible representation of $L_+^\uparrow \cup L_-^\uparrow$: unitary, $s_z \to s_z$, exchange the irreps. of $L_+^\uparrow$

$$\begin{aligned}
\psi &= \begin{pmatrix} \xi^a \\ \eta_{\dot{a}} \end{pmatrix}, \\
P\xi^a &= z_P \eta_a, \quad P\eta_a = z_P \xi^a, \\
z_P &= \pm 1, \ (P^2 = \mathbb{1}) \quad \text{or} \quad \pm i \ (P^2 = -\mathbb{1}).
\end{aligned} \tag{37}$$

2. T:

   (a) $TL(\boldsymbol{v}) = L(-\boldsymbol{v})T \neq L(\boldsymbol{v})P \Longrightarrow T \neq z_T\mathbb{1}$

   (b) Irreducible representation of $L_+^\uparrow \cup L_-^\downarrow$: anti-unitary, $s_z \to -s_z$, keep the irreps. of $L_+^\uparrow$

$$\begin{aligned} T\xi^a &= z_T g_{ab}\xi^{*b}, \quad T\eta_a = z_T g^{ab}\eta_b^*, \\ T^2 &= U(R_{\boldsymbol{n}}(2\pi)) = -\mathbb{1}, \\ z_T &= \pm i. \end{aligned} \tag{38}$$

3. C: anti-unitary, $s_z \to -s_z$, exchange the irreps. of $L_+^\uparrow$

$$\begin{aligned} C\xi^a &= z_C g^{ab}\eta_a^*, \quad C\eta_a = -z_C g_{ab}\xi^{*b} \\ z_C &= \pm 1, \pm i. \end{aligned} \tag{39}$$

Usual convention of a PTC-symmetric quantum field theory: $z_P = -z_T = -z_C = i$.

F. Dirac equation

1. Covariant equation for $\psi = (\xi^a, \eta_{\dot a})$,

2. Involving $p^{a\dot a} = p^0\mathbb{1} + \boldsymbol{p\sigma}$ and $p_{a\dot a} = p^0\mathbb{1} - \boldsymbol{p\sigma}^{\mathrm{tr}}$.

$$\begin{aligned} p^{a\dot a}\eta_{\dot a} &= m\xi^a, \\ p_{a\dot a}\xi^a &= m\eta_{\dot a}, \end{aligned} \tag{40}$$

or

$$\begin{aligned} (p^0 + \boldsymbol{p\sigma})\eta &= m\xi, \\ (p^0 - \boldsymbol{p\sigma})\xi &= m\eta. \end{aligned} \tag{41}$$

Dynamical role of the mass: coupling the two spinors

Elimination of one spinor:

$$(p^2 - m^2)\xi = (p^2 - m^2)\eta = 0. \tag{42}$$

Covariant form: $p_\mu = i\partial_\mu$

$$0 = (i\gamma_{ch}^\mu \partial_\mu - m)\psi$$

$$\gamma^0_{ch} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \boldsymbol{\gamma}_{ch} = \begin{pmatrix} 0 & -\boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \gamma^0_{st} = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \quad \boldsymbol{\gamma}_{st} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix},$$
$$\gamma^\mu_{ch} = U\gamma^\mu_{st}U^\dagger, \quad U = \frac{1}{\sqrt{2}}(1-\gamma^5\gamma^0) = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1}_2 & -\mathbb{1}_2 \\ \mathbb{1}_2 & \mathbb{1}_2 \end{pmatrix}, \tag{43}$$

Scalar Dirac matrix:

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3, \quad \gamma^5_{ch} = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} \tag{44}$$

Minimal coupling: $p_\mu \to p_\mu - eA_\mu$, $\partial_\mu \to \partial_\mu + ieA_\mu$, $(i\partial\!\!\!/ - eA\!\!\!/ - m)\psi = 0$

Discrete symmetries:

- $P\psi = U_P\psi_P$, $f_P(t,\boldsymbol{x}) = f(t,-\boldsymbol{x})$, $A_{\mu P}(t,\boldsymbol{x}) = (\phi(t,\boldsymbol{x}),\boldsymbol{A}(t,\boldsymbol{x}))_P = (\phi(t,-\boldsymbol{x}),-\boldsymbol{A}(t,-\boldsymbol{x}))$,

  $U_P = i\gamma^0$, $U_P\gamma^0U_p^{-1} = \gamma^0$, $U_P\boldsymbol{\gamma}U_P^{-1} = -\boldsymbol{\gamma}$, $(i\partial\!\!\!/ - eA\!\!\!/_P - m)\psi_P = 0$,

- $T\psi = U_T\bar{\psi}_T$, $f_T(t,\boldsymbol{x}) = f(-t,\boldsymbol{x})$, $A_{\mu T}(t,\boldsymbol{x}) = (\phi(t,\boldsymbol{x}),\boldsymbol{A}(t,\boldsymbol{x}))_P = (-\phi(-t,\boldsymbol{x}),\boldsymbol{A}(-t,\boldsymbol{x}))$,

  $U_T = -i\gamma^1\gamma^3\gamma^0$, $U_T\gamma^0U_T^{-1} = \gamma^{\mathrm{tr}0}$, $U_T\boldsymbol{\gamma}U_T^{-1} = -\boldsymbol{\gamma}^{\mathrm{tr}}$, $(i\partial\!\!\!/ - eA\!\!\!/_T - m)\psi_T = 0$,

- $C\psi = U_C\bar{\psi}$, $U_C = -i\gamma^2\gamma^0$, $U_C\gamma^\mu U_C^{-1} = -\gamma^{\mathrm{tr}\mu}$, $(i\partial\!\!\!/ + eA\!\!\!/ - m)\psi_C = 0$.

Lorentz transformations:

- Fundamental representation:

$$\xi^a \to A^a{}_b\xi^b, \quad \xi \to A\xi, \quad A(a,\boldsymbol{a}) = a\mathbb{1} + i\boldsymbol{a}\boldsymbol{\sigma} \tag{45}$$

- Complex conjugate representation:

$$\begin{aligned} A^*(a,\boldsymbol{a}) &= (a\mathbb{1} + i\boldsymbol{a}\boldsymbol{\sigma})^* = a^*\mathbb{1} - i\boldsymbol{a}^*\boldsymbol{\sigma}^*, \quad \boldsymbol{\sigma}^* = -\sigma_2\boldsymbol{\sigma}\sigma_2 \\ &= \sigma_2 A(a^*,\boldsymbol{a}^*)\sigma_2 \\ \eta_{\dot{a}} &\to A_{\dot{a}}{}^{\dot{b}}\eta_{\dot{b}} = (g_{aa'}g^{bb'}A^{a'}{}_{b'})^*\eta_{\dot{b}}, \qquad g = i\sigma_2 \\ \eta &\to i\sigma_2 A^*(a,\boldsymbol{a})i\sigma_2^{\mathrm{tr}}\eta = i\sigma_2\sigma_2 A(a^*,\boldsymbol{a}^*)\sigma_2 i\sigma_2^{\mathrm{tr}}\eta = A(a^*,\boldsymbol{a}^*)\eta. \end{aligned} \tag{46}$$

- Bi-spinors: $A_{\boldsymbol{n}}(\alpha+i\beta)$, $\boldsymbol{u} = \alpha\boldsymbol{n}$, $\boldsymbol{v} = \beta\boldsymbol{n}$,

  1. Lorentz covariance:

$$A_{\boldsymbol{n}}(\alpha+i\beta) = e^{-\frac{i}{2}(\alpha+i\beta)\boldsymbol{n}\boldsymbol{\sigma}}, \qquad (\alpha+i\beta)\boldsymbol{n}\boldsymbol{\sigma} = \boldsymbol{u}\boldsymbol{\sigma} + i\boldsymbol{v}\boldsymbol{\sigma} = \frac{1}{2}\omega_{\mu\nu}\sigma^{\mu\nu}$$

$$\omega_{\mu\nu} = \begin{pmatrix} 0 & v_1 & v_2 & v_3 \\ -v_1 & 0 & u_3 & -u_2 \\ -v_2 & -u_3 & 0 & u_1 \\ -v_3 & u_2 & -u_1 & 0 \end{pmatrix}, \quad \sigma^{0j}_{ch} = i\begin{pmatrix} \sigma_j & 0 \\ 0 & -\sigma_j \end{pmatrix}, \quad \sigma^{jk}_{ch} = \epsilon^{jk\ell}\begin{pmatrix} \sigma_\ell & 0 \\ 0 & \sigma_\ell \end{pmatrix}$$

$$\psi \to e^{-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}}\psi \tag{47}$$

2. Representation independent form:

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]. \tag{48}$$

3. Spin: generators of $SO(3)$ rotations:

$$S_j = \frac{1}{2}\epsilon_{jk\ell}\sigma_{k\ell}$$

$$(S_{ch})_j = \frac{1}{2}\epsilon_{jk\ell}\epsilon^{k\ell m}\begin{pmatrix} \sigma_m & 0 \\ 0 & \sigma_m \end{pmatrix} = \begin{pmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix} \tag{49}$$

TABLE I: Transformation properties of bilinears.

Bilinear	Special Lorentz tr.	Space inv.	Time inv.	Charge conj.
$S = \bar\psi\psi$	S	S	S	S
$P = \bar\psi\gamma^5\psi$	P	$-P$	P	P
$V^\mu = \bar\psi\gamma^\mu\psi = (V^0, \boldsymbol{V})$	$\omega^\mu{}_\nu V^\nu$	$(V^0, -\boldsymbol{V})$	$(-V^0, \boldsymbol{V})$	V^μ
$A^\mu = \bar\psi\gamma^5\gamma^\mu\psi$	$\omega^\mu{}_\nu V^\nu$	$(-V^0, \boldsymbol{V})$	$(-V^0, \boldsymbol{V})$	A^μ
$T^{\mu\nu} = \bar\psi\sigma^{\mu\nu}\psi = T^{\mu\nu}(\boldsymbol{u}, \boldsymbol{v})$	$\omega^\mu{}_\nu\omega^{\mu'}{}_{\nu'}T^{\nu\nu'}$	$T^{\mu\nu}(-\boldsymbol{u}, \boldsymbol{v})$	$T^{\mu\nu}(-\boldsymbol{u}, \boldsymbol{v})$	$T^{\mu\nu}(\boldsymbol{u}, \boldsymbol{v})$

G. Free particles

$$\psi^{(+)}(x) = e^{-ipx}u_{\boldsymbol{p}}, \quad \psi^{(-)}_p(x) = e^{ipx}v_{\boldsymbol{p}}, \qquad 0 = (\not p - m)u_{\boldsymbol{p}} = (\not p + m)v_{\boldsymbol{p}} \tag{50}$$

$p^0 = \omega_{\boldsymbol{p}} = \sqrt{m^2 + \boldsymbol{p}^2} \geq 0$, $p^2 = m^2c^2$

$\boldsymbol{p} \neq 0$: $p^\mu_0 = (mc, \boldsymbol{0})$,

$$0 = (\gamma^0 - 1)u_{\boldsymbol{0}} = (\gamma^0 + 1)v_{\boldsymbol{0}}$$

$$u_{\mathbf{0}}^{(1)} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}\phi^{(1)}\\0\end{pmatrix}, \quad u_{\mathbf{0}}^{(2)} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}\phi^{(2)}\\0\end{pmatrix},$$

$$v_{\mathbf{0}}^{(1)} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\\chi^{(1)}\end{pmatrix}, \quad v_{\mathbf{0}}^{(1)} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\\chi^{(2)}\end{pmatrix}. \tag{51}$$

$\boldsymbol{p} \neq 0$: $p^\mu = (\omega_{\boldsymbol{p}}, \boldsymbol{p})$, $(\not{p} \pm m)(\not{p} \mp m) = p^2 - m^2$, $\gamma^0 = \begin{pmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{pmatrix}$, $\boldsymbol{\gamma} = \begin{pmatrix}0 & \boldsymbol{\sigma}\\ -\boldsymbol{\sigma} & 0\end{pmatrix}$

$$u_{\boldsymbol{p}}^{(\alpha)} = \frac{\not{p} + m}{\sqrt{2m(m+\omega_p)}} u_{\mathbf{0}}^{(\alpha)} = \begin{pmatrix}\sqrt{\frac{m+\omega_{\boldsymbol{p}}}{2m}}\phi^{(\alpha)}\\ \frac{\boldsymbol{\sigma p}}{\sqrt{2m(m+\omega_{\boldsymbol{p}})}}\phi^{(\alpha)}\end{pmatrix}$$
$$v_{\boldsymbol{p}}^{(\alpha)} = \frac{-\not{p} + m}{\sqrt{2m(m+\omega_p)}} v_{\mathbf{0}}^{(\alpha)} = \begin{pmatrix}\frac{\boldsymbol{\sigma p}}{\sqrt{2m(m+\omega_{\boldsymbol{p}})}}\chi^{(\alpha)}\\ \sqrt{\frac{m+\omega_{\boldsymbol{p}}}{2m}}\chi^{(\alpha)}\end{pmatrix} \tag{52}$$

Normalization: $\gamma^0\gamma^\mu\gamma^0 = \gamma^{\mu\dagger}$, $u_{\mathbf{0}}^{(\alpha)\dagger}\gamma^\mu u_{\mathbf{0}}^{(\alpha)} = 0$, $u_{\mathbf{0}}^{(\alpha)\dagger}\gamma^\mu\gamma^\nu\gamma^\rho u_{\mathbf{0}}^{(\alpha)} = 0$

$$\begin{aligned}
\bar{u}_{\boldsymbol{p}}^{(\alpha)} u_{\boldsymbol{p}}^{(\beta)} &= \frac{u_{\mathbf{0}}^{(\alpha)\dagger}(\not{p}^\dagger + m)\gamma^0(\not{p} + m)u_{\mathbf{0}}^{(\beta)}}{2m(m+\omega_p)} = \frac{u_{\mathbf{0}}^{(\alpha)\dagger}\gamma^{02}(\not{p}^\dagger + m)\gamma^0(\not{p} + m)u_{\mathbf{0}}^{(\beta)}}{2m(m+\omega_p)}\\
&= \frac{\bar{u}_{\mathbf{0}}^{(\alpha)}(\not{p} + m)(\not{p} + m)u_{\mathbf{0}}^{(\beta)}}{2m(m+\omega_p)} = \frac{\bar{u}_{\mathbf{0}}^{(\alpha)}(m^2 + \not{p}^2 + 2m\not{p})u_{\mathbf{0}}^{(\beta)}}{2m(m+\omega_p)}\\
&= \frac{\bar{u}_{\mathbf{0}}^{(\alpha)}[m^2 + p^2 + 2m(\omega_p\gamma^0 - \boldsymbol{p\gamma})]u_{\mathbf{0}}^{(\beta)}}{2m(m+\omega_p)} = \frac{\bar{u}_{\mathbf{0}}^{(\alpha)}(m^2 + \omega_p^2 - \boldsymbol{p}^2 + 2m\omega_p)u_{\mathbf{0}}^{(\beta)}}{2m(m+\omega_p)}\\
&= \frac{\bar{u}_{\mathbf{0}}^{(\alpha)}(m^2 + m^2 + \boldsymbol{p}^2 - \boldsymbol{p}^2 + 2m\omega_p)u_{\mathbf{0}}^{(\beta)}}{2m(m+\omega_p)} = 1\\
\bar{u}_{\boldsymbol{p}}^{(\alpha)} u_{\boldsymbol{p}}^{(\beta)} &= -\bar{v}_{\boldsymbol{p}}^{(\alpha)} v_{\boldsymbol{p}}^{(\beta)} = \delta_{\alpha,\beta}, \qquad \bar{u}_{\boldsymbol{p}}^{(\alpha)} v_{\boldsymbol{p}}^{(\beta)} = \bar{v}_{\boldsymbol{p}}^{(\alpha)} u_{\boldsymbol{p}}^{(\beta)} = 0
\end{aligned} \tag{53}$$

Projection to positive and negative energy: momentum-dependence as for scalar particles

$$\begin{aligned}
P_+(\boldsymbol{p}) &= \sum_{\alpha=1}^{2} u_{\boldsymbol{p}}^{(\alpha)} \otimes \bar{u}_{\boldsymbol{p}}^{(\alpha)} = \frac{\not{p} + m}{\sqrt{2m(m+\omega_{\boldsymbol{p}})}}\frac{1+\gamma^0}{2}\frac{\not{p} + m}{\sqrt{2m(m+\omega_{\boldsymbol{p}})}} = \frac{\not{p} + m}{2m}\\
P_-(\boldsymbol{p}) &= -\sum_{\alpha=1}^{2} v_{\boldsymbol{p}}^{(\alpha)} \otimes \bar{v}_{\boldsymbol{p}}^{(\alpha)} = \frac{\not{p} - m}{\sqrt{2m(m+\omega_{\boldsymbol{p}})}}\frac{1-\gamma^0}{2}\frac{\not{p} - m}{\sqrt{2m(m+\omega_{\boldsymbol{p}})}} = \frac{m - \not{p}}{2m}
\end{aligned} \tag{54}$$

Bi-spinor: (2 spin) $\times$ ($\pm$ energy)

H. Helicity, chirality, Weyl and Majorana fermions

Bi-spinor: 8 real (4 complex) dimensional space: $8 = 4 + 4$

- **Helicity:**

  1. Rotation invariant spin projection: projection of the total angular momentum on the momentum,

$$
\begin{aligned}
h_{\boldsymbol{p}} &= \frac{\boldsymbol{J}\boldsymbol{p}}{|\boldsymbol{p}|}, \quad \boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S} = \boldsymbol{x} \times \boldsymbol{p} + \boldsymbol{S} \\
&= \frac{\boldsymbol{S}\boldsymbol{p}}{|\boldsymbol{p}|},
\end{aligned} \tag{55}
$$

  Dirac fermion: $h = \pm\frac{\hbar}{2}$.

  2. Conservation: $S_j = \frac{1}{2}\epsilon_{jk\ell}\sigma_{k\ell}$,

$$
\begin{aligned}
[\boldsymbol{\alpha}\boldsymbol{p} + \beta m, \boldsymbol{p}\boldsymbol{S}] &= \left[\boldsymbol{p}\begin{pmatrix} 0 & 1\!\!1_2 \\ 1\!\!1_2 & 0 \end{pmatrix}\begin{pmatrix} 0 & -\boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} + \beta m, \boldsymbol{p}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}\right] \\
&= p^j p^k \left[\begin{pmatrix} \sigma_j & 0 \\ 0 & -\sigma_j \end{pmatrix}, \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}\right] + mp^k \left[\begin{pmatrix} 0 & 1\!\!1_2 \\ 1\!\!1_2 & 0 \end{pmatrix}, \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}\right] \\
&= p^j p^k [\sigma_j, \sigma_k]\begin{pmatrix} 1\!\!1_2 & 0 \\ 0 & -1\!\!1_2 \end{pmatrix} + mp^k[1\!\!1_2, \sigma_k]\begin{pmatrix} 0 & 1\!\!1_2 \\ 1\!\!1_2 & 0 \end{pmatrix} = 0
\end{aligned} \tag{56}
$$

  3. Spatial rotation: invariant

  4. Lorentz boost: $\boldsymbol{p} = \frac{m\boldsymbol{v}}{\sqrt{1-\frac{v^2}{c^2}}}$,

     (a) $L_{-\lambda\boldsymbol{v}}$, $\lambda > 1$ such that $\boldsymbol{v}$ changes sign

     (b) $\boldsymbol{p} = p\boldsymbol{n}$, $\boldsymbol{S}\boldsymbol{n} = \frac{1}{2}n_j\epsilon_{jk\ell}\sigma^{k\ell} = \frac{1}{2}n_j\epsilon_{jk\ell}\frac{i}{2}[\gamma^k, \gamma^\ell] = \frac{i}{2}n_j\epsilon_{jk\ell}\gamma^k\gamma^\ell = \frac{i}{2}\boldsymbol{\gamma}(\boldsymbol{n} \times \boldsymbol{\gamma})$

     (c) Three vectors which are orthogonal to the boost velocity remain invariant

     (d) $\Longrightarrow \boldsymbol{S}\boldsymbol{n}$ invariant

     (e) $\boldsymbol{v}$ changes sign $\Longrightarrow h$ changes sign $\Longrightarrow$helicity is non-Lorentz invariant

     (f) A massive fermion can not be composed exclusively from a given helicity components.

- **Chirality:**

  1. Definition: Eigenvalue of $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, $\gamma^5_{ch} = \begin{pmatrix} 1\!\!1_2 & 0 \\ 0 & -1\!\!1_2 \end{pmatrix}$, $\gamma^5\psi = \chi\psi$, $\chi = \pm 1$

2. Projectors:

$$P_+ = R = \frac{1}{2}(\mathbb{1} + \gamma^5), \quad P_- = L = \frac{1}{2}(\mathbb{1} - \gamma^5), \tag{57}$$

3. Lorentz invariant, $\psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$, $\gamma^5_{ch}\xi = \xi$, $\gamma^5_{ch}\eta = -\eta$

4. Conserved by massless particles only,

$$\begin{aligned} (p^0 + \boldsymbol{p\sigma})\eta &= m\xi, \\ (p^0 - \boldsymbol{p\sigma})\xi &= m\eta. \end{aligned} \tag{58}$$

5. Right and left handed massless particles in the chiral representation:

   (a)

$$\begin{aligned} \not{p}\psi &= (\omega_p\gamma^0 - \boldsymbol{p\gamma})\psi = 0, \quad \boldsymbol{p\gamma}\psi = \omega_p\gamma^0\psi \\ \boldsymbol{p}\gamma^0\boldsymbol{\gamma}\psi &= \boldsymbol{p}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma} \end{pmatrix}\psi = \boldsymbol{pS}\begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}\psi = |\boldsymbol{p}|h\begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}\psi \\ &= |\boldsymbol{p}|h\chi\psi = \omega_p\psi = |\boldsymbol{p}|\psi \end{aligned} \tag{59}$$

   (b) $h = \chi$

6. Weyl fermions: ξ and η, respect no space inversion.

   (a) Lagrangian:

$$L_\eta = \eta^\dagger(i\partial_0 - i\boldsymbol{\nabla\sigma})\eta, \quad L_\xi = \xi^\dagger(i\partial_0 + i\boldsymbol{\nabla\sigma})\xi. \tag{60}$$

   (b) "Neutrino equations": $(p^0 + \boldsymbol{p\sigma})\eta = (p^0 - \boldsymbol{p\sigma})\xi = 0$

   (c) ν, seen in the mirror does not exist in Nature.

TABLE II: Invariance properties of the splitting of the bi-spinor space.

Invariance	Helicity	Chirality	Majorana fermion
$L_+^\uparrow$	×	✓	✓
Time evolution of a free particle	✓	$m \neq 0$: ×; $m = 0$: ✓	✓

- **Majorana fermion:**

1. Definition: The basis transformation

$$U_M = \frac{1}{2}\begin{pmatrix} 1 + \sigma_2 & i(\sigma_2 - 1) \\ i(1 - \sigma_2) & 1 + \sigma_2 \end{pmatrix} \tag{61}$$

   makes

(a) $\gamma_M^\mu = U\gamma_{ch}^\mu U^\dagger$ imaginary,

$$\gamma_M^0 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \quad \gamma_M^1 = \begin{pmatrix} i\sigma_1 & 0 \\ 0 & i\sigma_1 \end{pmatrix}, \quad \gamma_M^2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_1 & 0 \end{pmatrix}, \quad \gamma_M^3 = \begin{pmatrix} i\sigma_3 & 0 \\ 0 & i\sigma_3 \end{pmatrix}, \tag{62}$$

(b) $(i\gamma^\mu\partial_\mu - m)\psi = 0$ real

(c) phase of ψ preserved

2. Majorana-Weyl fermion:

(a) Realness of a Dirac fermion, $\psi^* = \psi$, is not a Lorentz invariant condition.

(b) But the bi-spinor

$$\begin{aligned} \psi_{Mch} &= \begin{pmatrix} \xi \\ -i\sigma_2\xi^* \end{pmatrix} \\ \psi_M &= \frac{1}{2}\begin{pmatrix} 1+\sigma_2 & i(\sigma_2-1) \\ i(1-\sigma_2) & 1+\sigma_2 \end{pmatrix}\begin{pmatrix} \xi \\ -i\sigma_2\xi^* \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} (1+\sigma_2)\xi + (\sigma_2-1)\sigma_2\xi^* \\ i(1-\sigma_2)\xi - i(1+\sigma_2)\sigma_2\xi^* \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (1+\sigma_2)\xi + [(1+\sigma_2)\xi]^* \\ i(1-\sigma_2)\xi + [i(1-\sigma_2)\xi]^* \end{pmatrix} \end{aligned} \tag{63}$$

is real.

(c) Mas generation?

i. Lagrangian:

$$\begin{aligned} L_M &= \frac{1}{2}\bar\psi_M(i\partial\!\!\!/ - m)\psi_M \\ &= \frac{1}{2}[\xi^\dagger i(\partial_0 - \boldsymbol{\nabla\sigma})\xi - m\xi^{\rm tr} i\sigma_2\xi] + c.c. \end{aligned} \tag{64}$$

ii. E.O.M.:

$$(\partial_0 - \boldsymbol{\nabla\sigma})\xi - m\sigma_2\xi^* = 0. \tag{65}$$

iii. Are the neutrinos massive?

3. Twistors: space-time coordinates form spinors

(a) Observables are bosonic tensor operator with integer spin.

- Observed coordinates belong to $\ell = 1$.
- can the elementary, microscopic representation be simpler, say $\ell = \frac{1}{2}$?

(b) Adjoint representation

$$x^\mu \leftrightarrow X^{a\dot{a}} = x^0 \mathbb{1} + \boldsymbol{x\sigma} = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} \leftrightarrow \nu^a \tilde{\nu}^{\dot{a}} \tag{66}$$

(c) $d = 4 \Longrightarrow X^\dagger = X \Longrightarrow \tilde{\nu} = \nu^*$

(d) Twistor coordinate: $X^{a\dot{a}}(x) = \nu^a \nu^{*\dot{a}}$.

(e) However $\nu \to e^{i\alpha}\nu$ is a symmetry $\Longrightarrow d \to d-1 = 3$ can be represented.

(f) $\mathrm{Rank}(\nu^a \nu^{*\dot{a}}) = 1 \Longrightarrow \det X = x^\mu x_\mu = 0$.

(g) Twistors are available only for light-like vectors.

(h) General vectors might be represented by a bi-twistor, $(\nu^a, \tau_b) \Longrightarrow$simplicity is lost

I. Non-relativistic limit

$$\frac{i}{c}\partial_t \psi = \left[\boldsymbol{\alpha}(-i\boldsymbol{\nabla} - \frac{e}{c}\boldsymbol{A}) + \beta mc + \frac{e}{c}A_0\right]\psi = \left[\boldsymbol{\alpha\pi} + \beta mc + \frac{e}{c}A_0\right]\psi, \quad \boldsymbol{\pi} = \boldsymbol{p} - \frac{e}{c}\boldsymbol{A} \tag{67}$$

Pauli's equation: Standard representation, $\psi = (\phi, \chi)$,

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \quad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \\ i\partial_t \phi &= c\boldsymbol{\sigma\pi}\chi + (eA_0 + mc^2)\phi \\ i\partial_t \chi &= c\boldsymbol{\sigma\pi}\phi + (eA_0 - mc^2)\chi \end{aligned} \tag{68}$$

Separation of the rest mass: $\Phi = e^{imc^2t}\phi$ and $X = e^{imc^2t}\chi$

$$\begin{aligned} i\partial_t \Phi &= c\boldsymbol{\sigma\pi}X + eA_0\Phi \\ i\partial_t X &= c\boldsymbol{\sigma\pi}\Phi + (eA_0 - 2mc^2)X \\ X &= \frac{\boldsymbol{\sigma\pi}}{2mc}\Phi \quad \leftarrow \quad \partial_t, \quad eA_0 \ll mc^2 \\ i\partial_t \Phi &= \left[\frac{(\boldsymbol{\sigma\pi})^2}{2m} + eA_0\right]\Phi, \end{aligned} \tag{69}$$

Summary: Use Schrödinger's equations with $\boldsymbol{p} \to \boldsymbol{\sigma\pi}$.

$$\begin{aligned} (\boldsymbol{\sigma\pi})^2 &= \sigma_j\sigma_k\pi_j\pi_k = \frac{1}{4}([\sigma_j,\sigma_k] + \{\sigma_j,\sigma_k\})([\pi_j,\pi_k] + \{\pi_j,\pi_k\}) \\ &= \frac{1}{4}(\{\sigma_j,\sigma_k\}\{\pi_j,\pi_k\} + [\sigma_j,\sigma_k][\pi_j,\pi_k]), \quad \sigma_j\sigma_k = \delta_{jk} + i\epsilon_{jk\ell}\sigma_\ell \\ &= \boldsymbol{\pi}^2 + \frac{i}{2}\epsilon_{jk\ell}\sigma_\ell[-i\nabla_j - \frac{e}{c}A_j, -i\nabla_k - \frac{e}{c}A_k] \end{aligned}$$

$$= \boldsymbol{\pi}^2 + \frac{i}{2}\epsilon_{jk\ell}\sigma_\ell \frac{e}{c} i([\nabla_j, A_k] - [\nabla_k, A_j]) = \boldsymbol{\pi}^2 + \frac{i}{2}\epsilon_{jk\ell}\sigma_\ell \frac{e}{c} i(\nabla_j A_k - \nabla_k A_j)$$
$$([\nabla, f]\psi = \nabla(f\psi) - f\nabla\psi = (\nabla f))$$
$$= \boldsymbol{\pi}^2 - \frac{e}{c}\boldsymbol{B}\boldsymbol{\sigma}, \quad \boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A} \tag{70}$$

Pauli's equation:

$$i\partial_t\Phi = \left[\frac{(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A})^2}{2m} - \frac{e}{2mc}\boldsymbol{\sigma}\boldsymbol{B} + eA_0\right]\Phi. \tag{71}$$

Problems:

1. Irrealistic velocity:

   (a) Representations of the time evolution:

      i. Schrödinger: $i\partial_t|\psi(t)\rangle_S = H|\psi(t)\rangle_S$, $i\partial_t A_S = 0$, $|\psi(t)\rangle_S = e^{-i(t-t_i)H}|\psi(t_i)\rangle_S$

      ii. Heisenberg: $|\psi(t)\rangle_H = e^{i(t-t_i)H}|\psi(t)\rangle_S = |\psi(t_i)\rangle_S$,

$$\begin{aligned}
\langle\psi(t)|A|\psi(t)\rangle_H &= \underbrace{\langle\psi(t_i)|e^{i(t-t_i)H}}_{\langle\psi(t)|_S}\underbrace{A}_{A_S}\underbrace{e^{-i(t-t_i)H}|\psi(t_i)\rangle_S}_{|\psi(t)\rangle_S} \\
&= \underbrace{\langle\psi(t_i)|}_{\langle\psi|_H}\underbrace{e^{i(t-t_i)H}Ae^{-i(t-t_i)H}}_{A_H(t)}\underbrace{|\psi(t_i)\rangle_S}_{|\psi\rangle_H} \\
A_H(t) &= e^{i(t-t_i)H}A_S e^{-i(t-t_i)H} \\
i\partial_t A_H(t) &= [A_H, H], \quad i\partial_t|\psi(t)\rangle_H = 0
\end{aligned} \tag{72}$$

   (b) Dirac equation:

$$\frac{i}{c}\partial_t\boldsymbol{x} = [\boldsymbol{x}, \boldsymbol{\alpha}\boldsymbol{p} + \beta mc^2] = i\boldsymbol{\alpha}, \tag{73}$$

   $\alpha_j\psi = \pm\psi$, the particle moves with the speed of light.

   (c) Zitterbewegung: interference between positive and negative energy solutions of the Dirac equation.

   (d) Heisenberg uncertainty principle: good knowledge of $\boldsymbol{x}$ $\Longrightarrow$large momentum fluctuations.

2. Higher order time derivatives $\Longrightarrow$instabilities

Foldy-Wouthuysen transformation:

1. Basis transformation $|\psi\rangle \to S|\psi\rangle$ to decouple the small and the large components

$$S = e^{-\frac{\gamma\boldsymbol{p}}{|\boldsymbol{p}|}\theta} = \mathbb{1}\cos\theta + \frac{\gamma\boldsymbol{p}}{|\boldsymbol{p}|}\sin\theta, \qquad ((\gamma\boldsymbol{p})^2 = -\boldsymbol{p}^2)$$

$$\begin{aligned}
S(\boldsymbol{\alpha p}+\beta m)S^{-1} &= \left(\mathbb{1}\cos\theta+\frac{\boldsymbol{\gamma p}}{|\boldsymbol{p}|}\sin\theta\right)(\boldsymbol{\alpha p}+\beta m)\left(\mathbb{1}\cos\theta-\frac{\boldsymbol{\gamma p}}{|\boldsymbol{p}|}\sin\theta\right)\\
&= (\boldsymbol{\alpha p}+\beta m)\left(\mathbb{1}\cos\theta-\frac{\boldsymbol{\gamma p}}{|\boldsymbol{p}|}\sin\theta\right)^2,\ (p^jp^k\{\gamma^j,\alpha^k\}=p^jp^k\gamma^0[\gamma^j,\gamma^k]=0)\\
&= (\boldsymbol{\alpha p}+\beta m)\left(\mathbb{1}(\cos^2\theta-\sin^2\theta)-2\frac{\boldsymbol{\gamma p}}{|\boldsymbol{p}|}\cos\theta\sin\theta\right)\\
&= (\boldsymbol{\alpha p}+\beta m)\left(\mathbb{1}\cos2\theta-\frac{\boldsymbol{\gamma p}}{|\boldsymbol{p}|}\sin2\theta\right)\\
&= \boldsymbol{\alpha p}\left(\cos2\theta-\frac{m}{|\boldsymbol{p}|}\sin2\theta\right)+\beta(m\cos2\theta+|\boldsymbol{p}|\sin2\theta),\\
\sin2\theta &= \frac{|\boldsymbol{p}|}{\omega_p},\quad \cos2\theta=\frac{m}{\omega_p},\quad \tan2\theta=\frac{|\boldsymbol{p}|}{m},\quad H_{FW}=\beta\omega_p.
\end{aligned}\tag{74}$$

2. Projectors:

$$\begin{aligned}
SP_\pm(\boldsymbol{p})S^{-1} &= \left(\mathbb{1}\cos\theta+\frac{\boldsymbol{\gamma p}}{|\boldsymbol{p}|}\sin\theta\right)\frac{m\pm(\beta\omega_p-\boldsymbol{\gamma p})}{2m}\left(\mathbb{1}\cos\theta-\frac{\boldsymbol{\gamma p}}{|\boldsymbol{p}|}\sin\theta\right)\\
&= \frac{m\mp\boldsymbol{\gamma p}}{2m}\pm\frac{\beta\omega_p}{2m}\left(\mathbb{1}\cos\theta-\frac{\boldsymbol{\gamma p}}{|\boldsymbol{p}|}\sin\theta\right)^2,\quad \cos^2\theta-\sin^2\theta=\frac{m}{\omega_p}\\
&= \frac{m\mp\boldsymbol{\gamma p}}{2m}\pm\frac{\beta\omega_p}{2m}\left(\mathbb{1}\frac{m}{\omega_p}-\frac{\boldsymbol{\gamma p}}{|\boldsymbol{p}|}\frac{|\boldsymbol{p}|}{\omega_p}\right)\\
&= \frac{\mathbb{1}\pm\beta}{2}
\end{aligned}\tag{75}$$

3. Coordinate operator: $\boldsymbol{x}=i\boldsymbol{\nabla}_p$

$$\begin{aligned}
\boldsymbol{x}_{FW} &= S\boldsymbol{x}S^{-1}\\
&= \left(\mathbb{1}\cos\theta+\frac{\boldsymbol{\gamma p}}{|\boldsymbol{p}|}\sin\theta\right)\boldsymbol{x}\left(\mathbb{1}\cos\theta-\frac{\boldsymbol{\gamma p}}{|\boldsymbol{p}|}\sin\theta\right)\\
&= \boldsymbol{x}+i\boldsymbol{a},\quad \boldsymbol{a}=-\boldsymbol{\nabla}_p\left(\frac{\boldsymbol{\gamma p}}{|\boldsymbol{p}|}\sin\theta\right)
\end{aligned}\tag{76}$$

4. Velocity operator:

$$\frac{1}{c}\partial_t\boldsymbol{x}=-i[\boldsymbol{x},\beta\omega_p]=\beta\boldsymbol{\nabla}_p\omega_p=\beta\boldsymbol{v}_{group}\tag{77}$$

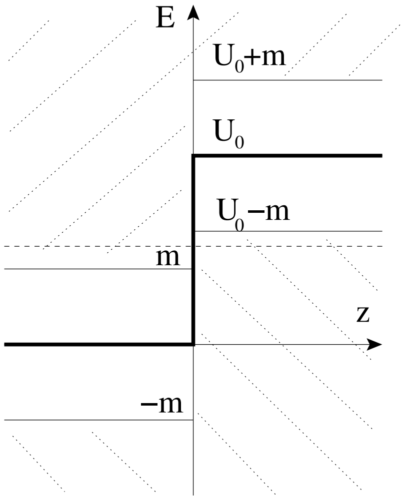
II. PARADOXES

A. Step potential

1+1 dimensions, $U(z)=U_0\Theta(z)$, $U_0>2m$

Klein-Gordon equation:

$$\begin{aligned}
\psi(t,z) &= \chi(z)e^{-itE}\\
0 &= [(E-U(z))^2+\nabla_z^2-m^2]\chi(z)
\end{aligned}$$



$$\begin{aligned} \chi(z) &= \Theta(-z)[\chi_i(z)+\chi_r(z)] + \Theta(z)\chi_t(z) \\ \chi_i(z) &= e^{ipz}, \quad \chi_r(z) = be^{-ipz}, \quad \chi_t(z) = de^{ip'z}, \quad p = \sqrt{E^2-m^2}, \quad p' = \sqrt{(E-U_0)^2-m^2} \end{aligned} \tag{78}$$

Matching conditions:

$$\begin{aligned} \chi(0^-) &= \chi(0^+) \quad \to \quad 1+b=d \\ \chi'(0^-) &= \chi'(0^+) \quad \to \quad 1-b = d\xi = (1+b)\xi, \quad \xi = \frac{p'}{p} \\ 1-\xi &= b(1+\xi), \quad b = \frac{1-\xi}{1+\xi}, \quad d = 1+b = \frac{2}{1+\xi} \end{aligned} \tag{79}$$

$$\begin{aligned} j^z &= \frac{1}{2im}(\chi^*\nabla_z\chi - \nabla_z\chi^*\chi), \quad j_i^z(0) = p, \quad j_r^z(0) = -|b|^2p, \quad j_t^z(0) = |d|^2p' \\ R &= |b|^2 = \frac{(1-\xi)^2}{(1+\xi)^2}, \quad T = |d|^2\xi = \frac{4\xi}{(1+\xi)^2} \end{aligned} \tag{80}$$

1. Current conservation: $j_i^z(0) + j_r^z(0) = j_t^z(0) \Longrightarrow R+T=1$.

2. $2m < U_0$, $m < E < U_0 - m \Longrightarrow \xi > 0$, $0 \le R, T \le 1$.

3. Particle-anti particle mixing:

$$j_0(z) = \frac{i}{2m}\phi^*\overleftrightarrow{\partial_0}\phi = \frac{1}{m}\begin{cases} E>0 & z<0 \\ E-U<0 & z>0 \end{cases} \tag{81}$$

Dirac equation:

$$\begin{aligned} \psi(t,z) &= \chi(z)e^{-itE} \\ 0 &= [\gamma^0(E-U(z)) + i\gamma^z\nabla_z - m]\chi(z) = 0 \end{aligned}$$

$$
\begin{aligned}
u_{\boldsymbol{p}}^{(\alpha)} &= \frac{\not{p}+m}{\sqrt{2m(m+\omega_p)}} u_{\boldsymbol{0}}^{(\alpha)} = \begin{pmatrix} \sqrt{\frac{m+\omega_{\boldsymbol{p}}}{2m}}\phi^{(\alpha)} \\ \frac{\boldsymbol{\sigma p}}{\sqrt{2m(m+\omega_{\boldsymbol{p}})}}\phi^{(\alpha)} \end{pmatrix} \\
v_{\boldsymbol{p}}^{(\alpha)} &= \frac{-\not{p}+m}{\sqrt{2m(m+\omega_p)}} v_{\boldsymbol{0}}^{(\alpha)} = \begin{pmatrix} \frac{\boldsymbol{\sigma p}}{\sqrt{2m(m+\omega_{\boldsymbol{p}})}}\chi^{(\alpha)} \\ \sqrt{\frac{m+\omega_{\boldsymbol{p}}}{2m}}\chi^{(\alpha)} \end{pmatrix} \\
\chi(z) &= \Theta(-z)[\chi_i(z)+\chi_r(z)] + \Theta(z)\chi_t(z) \\
\chi_i(z) &= e^{ipz}\begin{pmatrix} 1 \\ 0 \\ \frac{p}{m+E} \\ 0 \end{pmatrix}, \quad E^2 = m^2+p^2 \\
\chi_r(z) &= be^{-ipz}\begin{pmatrix} 1 \\ 0 \\ -\frac{p}{m+E} \\ 0 \end{pmatrix} + b'e^{-ipz}\begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{p}{m+E} \end{pmatrix} \\
\chi_t(z) &= de^{ip'z}\begin{pmatrix} 1 \\ 0 \\ \frac{p'}{m+E-U_0} \\ 0 \end{pmatrix} + d'e^{ip'z}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -\frac{p'}{m+E-U_0} \end{pmatrix}, \quad (E-U_0)^2 = m^2+p'^2
\end{aligned} \tag{82}
$$

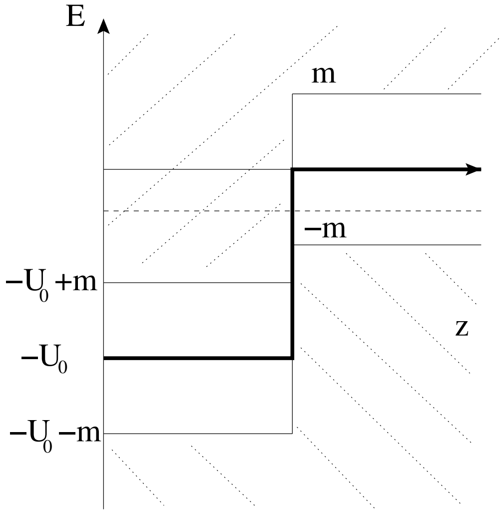
Spin independent potential (no spin flip): $b' = d' = 0$.

Matching condition:

$$
\begin{aligned}
\int_{-\epsilon}^{\epsilon} dz \nabla_z \chi(z) &= i\int_{-\epsilon}^{\epsilon} dz \gamma^z [m - \gamma^0(E-U(z))]\chi(z) \\
\mathrm{Disc}\chi(0) &= i\gamma^z \lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} dz[m-\gamma^0(E-U(z))]\chi(z) = 0 \\
\chi_1(0^-) &= \chi_1(0^+) \quad \to \quad 1+b = d \\
\chi_3(0^-) &= \chi_3(0^+) \quad \to \quad (1-b)\frac{p}{m+E} = d\frac{p'}{m+E-U_0}, \quad 1-b = d\xi, \quad \xi = \frac{p'}{p}\underbrace{\frac{m+E}{m+E-U_0}}_{\text{spin}} \\
b &= \frac{1-\xi}{1+\xi}, \quad d = \frac{2}{1+\xi}.
\end{aligned} \tag{83}
$$

Reflection and transmission coefficients:

$$
\begin{aligned}
j^z &= \bar{\psi}\gamma^z\psi = \psi^\dagger\gamma^0\gamma^z\psi = \chi^\dagger \begin{pmatrix} 0 & \sigma^z \\ \sigma^z & 0 \end{pmatrix}\chi \\
R &= -\frac{j_r}{j_i}, \quad T = \frac{j_t}{j_i} \\
j_i &= 2\frac{p}{m+E}, \quad j_r = -2|b|^2\frac{p}{m+E}, \quad j_t = 2|d|^2\frac{p'}{m+E-U_0}
\end{aligned}
$$



$$
\begin{aligned}
R &= |b|^2, \quad T = |d|^2 \frac{p'}{p} \frac{m+E-U_0}{m+E} \\
j_i(0)+j_r(0) &= \frac{2(1-|b|^2)p}{m+E} = \frac{2(1-|b|^2)p'}{\xi(m+E-U_0)} = j_t(0)\frac{1-|b|^2}{\xi|d|^2} = j_t(0) \quad \rightarrow \quad R+T=1
\end{aligned} \tag{84}
$$

Klein paradox: $m < E < U_0 - m \Longrightarrow R > 1$ and $T < 0$ (spin effect).

B. Spherical potential well (Oppenheimer-Schiff-Snyder effect)

Klein-Gordon equation with a spherical well, $U(r) = -U_0\Theta(R-r)$:

$$
\begin{aligned}
0 &= [(\partial_0 + iU(r))^2 - \Delta + m^2]\phi(x), \quad \phi_{lm}(x) = \eta_\ell(r) Y^\ell_m(\theta,\phi) e^{-itE} \\
0 &= \left[(E-U(r))^2 + \frac{1}{r^2}\partial_r r^2 \partial_r - \frac{l(l+1)}{r^2} - m^2\right]\eta_\ell(r), \quad \eta(r) = \frac{u(r)}{r} \\
0 &= \left[\partial_r^2 - \frac{l(l+1)}{r^2} + (E-U(r))^2 - m^2\right] u_\ell(r) = 0
\end{aligned} \tag{85}
$$

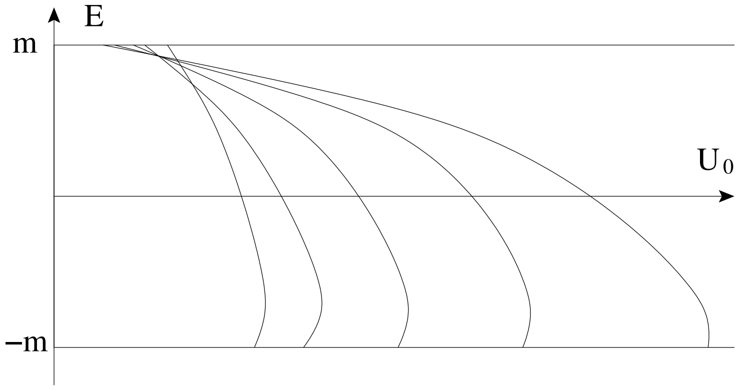
$-m < E < m$ and $E > m - U_0$, $\ell = 0$:

$$
\begin{aligned}
u_0'' &= [m^2 - (E-U(r))^2] u_0 \\
u_0 &= \Theta(R-r)\sin\kappa r + \Theta(r-R) a e^{-kr}, \quad \kappa = \sqrt{(E+U_0)^2 - m^2}, \quad k = \sqrt{m^2 - E^2} \\
\sin\kappa R &= a e^{-kR}, \quad \kappa\cos\kappa R = -k a e^{-kR} \\
\tan R\kappa &= -\frac{\kappa}{k} \quad \rightarrow \quad \tan R\sqrt{(E+U_0)^2 - m^2} = -\sqrt{\frac{(E+U_0)^2 - m^2}{m^2 - E^2}}
\end{aligned} \tag{86}
$$

1. Potential attracts both particle and anti-particle

2. Bound states acquire imaginary energy when meet: $E \to \pm iE$

3. Bound states disappear from the physical state for large U_0


E
m
U 0
−m