Special relativity

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Video:

https://drive.google.com/drive/folders/1VpbWufW2YU_XwDJYz4Z7p_VxHpFREVHO?usp=sharing

Contents

I. PHYSICAL LAWS AS THE FUNCTIONS OF THE SCALE OF OBSERVATION

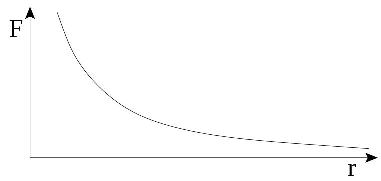
Newton's equation $F = ma$ is modified at $v \sim c = 2.9979 \cdot 10^8 m/s$ (speed of light)

Problem: our intuition belongs to $\frac{v}{c} \ll 1$

The observed quantities and the physical laws depend on the scale of observation

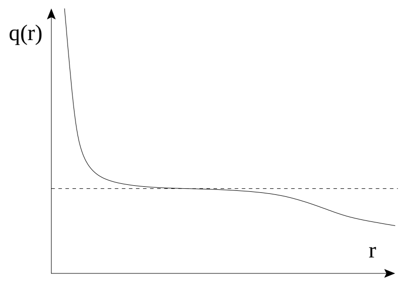
Scales: L, T, and M

Charge:



$$F_C(r) = \frac{qq'}{r^2} \neq F_{phys}(r) = \frac{q(r)q'}{r^2}$$

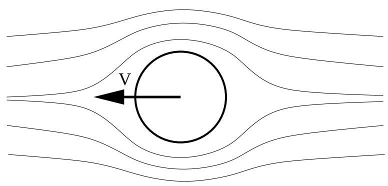
Vacuum polarization: $q \to q(r)$



Physical constant $\equiv$ plateau

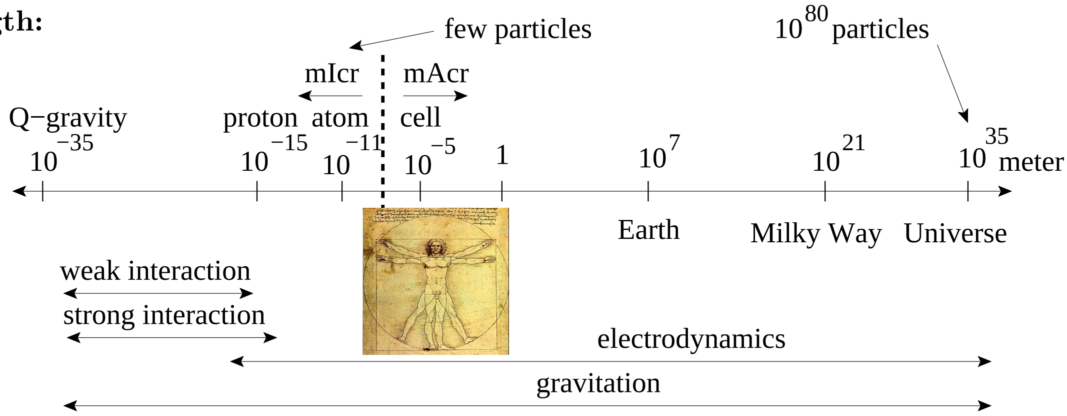
Mass:

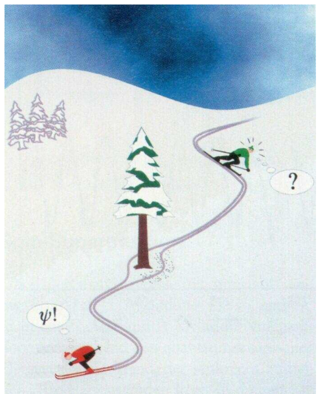
$$\begin{aligned} E(v) &= E(v_0) + (v - v_0)\frac{dE(v_0)}{dv} \\ &+ \frac{(v - v_0)^2}{2}\frac{d^2E(v_0)}{dv^2} + \cdots \\ M(v_0) &= \frac{d^2E(v_0)}{dv^2} \end{aligned}$$



Interactions with the environment $\implies$ effective parameters: $q \to q(r)$, $M \to M(v)$, etc.

Length:





Quantum mechanics:

- The Reality can not fully be known
- Quantum mechanics:

  Optimized and consistent treatment of partial information
- Quantum state: a list of virtual realities

  like a phone book: name $\leftrightarrow$virtual reality

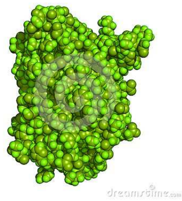
  tel. number$\leftrightarrow$probability

Life:

- elementary unit: protein
- living forms consist of 4500 proteins
- Origin: primordial soup?

  4^{165} possible RNA chains

  $M = 2 \times 10^{77} kg \sim M_{Universe} \times 10^{25}$

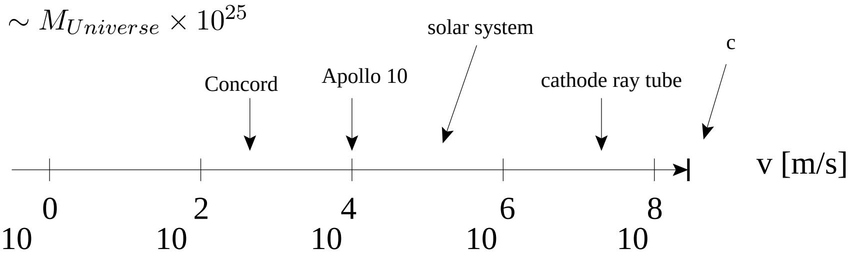


450 randomly chosen proteins

at the quantum-classical border

Quantum origin of life?

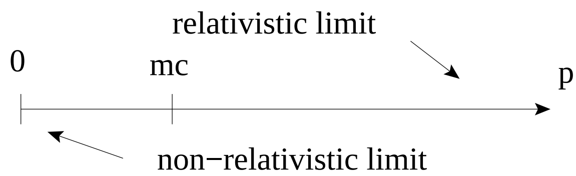
Velocity:



Orders of magnitudes in free fall: $v = 9.8t[MKS]$,

after 1 year $v = 9.8 \times 365 \times 24 \times 3600 \approx 3 \times 10^8 m/s$

Momentum:



II. A CONFLICT AND ITS SOLUTION

A. Reference frame and Galilean symmetry

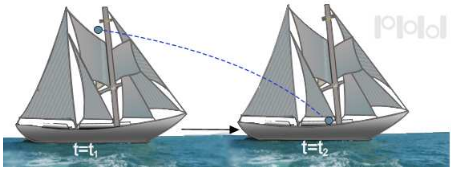
Reference frame: free motion = constant velocity, $S : (t, \boldsymbol{x})$, $\boldsymbol{x} = t\boldsymbol{v} + \boldsymbol{x}_0$.

Galilean relativity:

- a generalization of the dynamics of a free particle to the interactive case
- The mechanical laws are identical in each reference frame.

Example:

An object, falling freely from the the mast, ends up at the bottom of the mast on a ship which moves with constant velocity.



Symmetry transformations of Newton's equation: $m\frac{d^2\boldsymbol{x}}{dt^2} = 0$: (Galilean transformations)

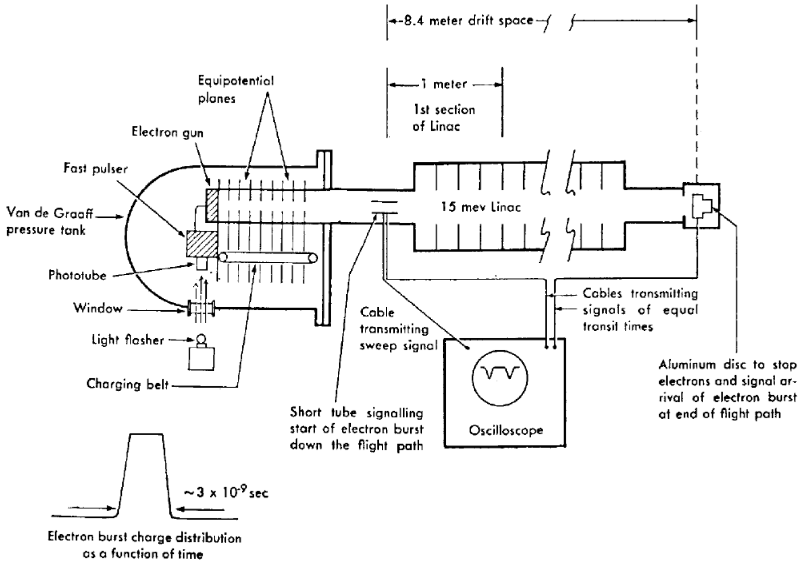
$$t \rightarrow t' = t \quad \leftarrow \quad \text{absolute time}$$
$$\boldsymbol{x} \rightarrow \boldsymbol{x'} = R\boldsymbol{x} - t\boldsymbol{v} + \boldsymbol{x}_0$$

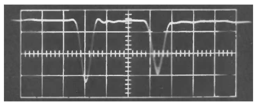
Different reference frames are connected by Galilean transformations.

Addition of the velocity: $\dot{\boldsymbol{x}} \rightarrow \dot{\boldsymbol{x}}' = \frac{d}{dt}(\boldsymbol{x} - t\boldsymbol{v} + \boldsymbol{x}_0) = \dot{\boldsymbol{x}} - \boldsymbol{v}$ (absolute time)

B. Limiting velocity

W. Bertozzi, Am. J. Phys. **32** 551 (1964)





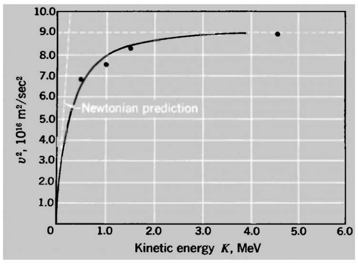
$\leftrightarrow 10^{-8}s$

Energy $K[MeV]$	Time of flight$t[10^{-8}s]$	Velocity $v[10^8 m/s]$	Velocity square $v^2[10^{16}m^2/s^2]$
0.5	3.23	2.60	6.8
1.0	3.08	2.73	7.5
1.5	2.92	2.88	8.3
4.5	2.84	2.96	8.8
15	2.80	3.00	9.0

as the energy increases like $E \to 30E$

energy conservation requires $K \to 30K$

Newton's equation: $v^2 \to 30v^2$?



Limiting velocity: $v < c$ (not enough energy to accelerate beyond c)

C. Particle or wave?

XIX.-th century physics: nature of light

- Thomas Young (1801-04): measurement of interference
- Augustin-Jean Fresnel (1818): explication of interference, diffraction, polarization
- James Clerk Maxwell (1861): electrodynamics

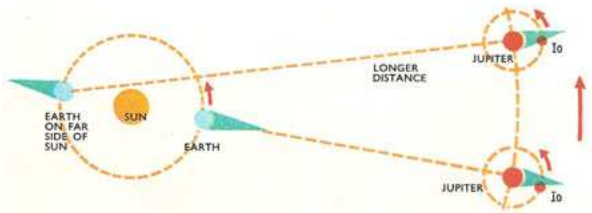
Wave: diffraction, interference, polarization $\leftrightarrow$ Particle: energy-momentum

Mechanical models until 1850 but the speed of light is too high for a particle

Speed of light :

Ole Roemer (1675): eclipse of Jupiter's moons varies in time

$c \approx 2 \times 10^8\ m/s$

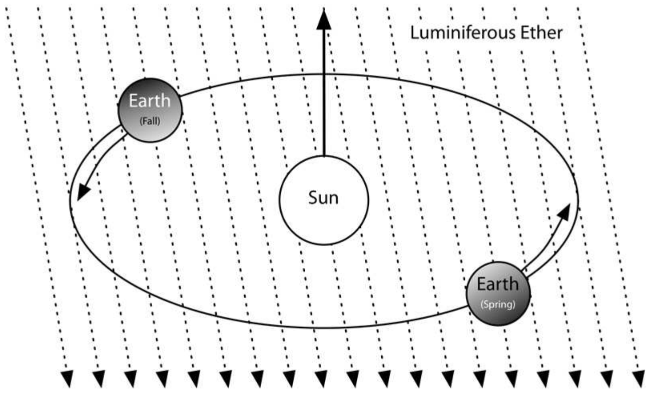


Wave or particle?

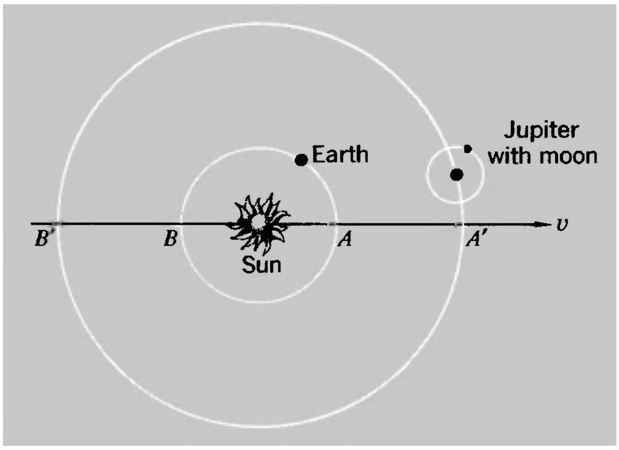
Particle: $v_{part} = v_{source} + v_{light}$,

Wave: v_{wave} is fixed within the medium

Ether hypothesis $\to$



How to find our velocity with respect to the ether?



Maxwell: 1 Jupiter year=12 Earth years,

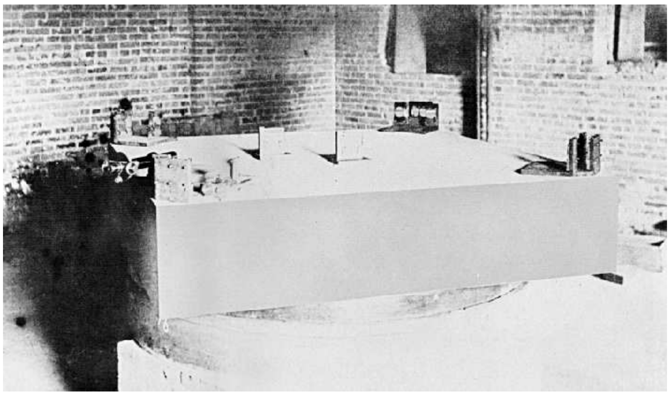
two measurements, separated by 6 tears

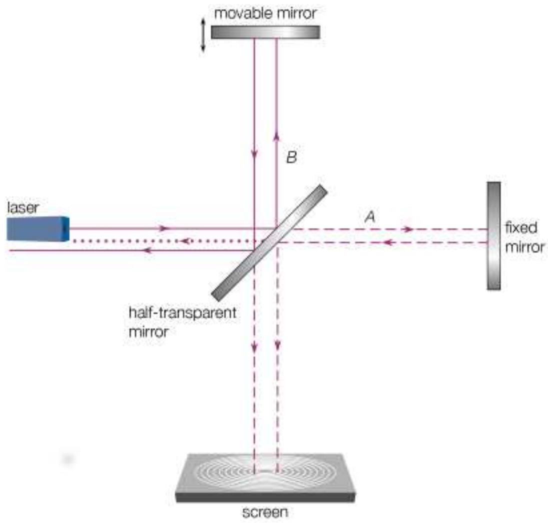
$$
\begin{aligned}
t_A^{écl.} &= \frac{\ell}{c+v_{sol}}, \qquad t_B^{écl.} = \frac{\ell}{c-v_{sol}} \\
t_B^{écl.} - t_A^{écl.} &= \frac{2\ell v_{sol}}{c^2 - v_{sol}^2} = \frac{2\ell v_{sol}}{(c+v_{sol})(c-v_{sol})} \\
&\approx \frac{2\ell v_{sol}}{c^2} = \underbrace{t_0}_{16min} \frac{2v_{sol}}{c}
\end{aligned}
$$

The problem is the repetition of the same experience 6 years later

D. Propagation of the light

Michelson (1881):





$$
t_{\|} = \frac{\ell_{\|}}{c-v} + \frac{\ell_{\|}}{c+v} = \frac{2c\ell_{\|}}{c^2-v^2} = \frac{\frac{2\ell_{\|}}{c}}{1-\frac{v^2}{c^2}}
$$

$$
\begin{aligned}
\left(\frac{ct_\perp}{2}\right)^2 &= \ell_\perp^2 + \left(\frac{vt_\perp}{2}\right)^2, \quad t_\perp^2(c^2-v^2) = 4\ell_\perp^2, \quad t_\perp = \frac{2\ell_\perp}{\sqrt{c^2-v^2}} \\
\Delta t(\ell_{\|}, \ell_\perp) &= t_{\|} - t_\perp = \frac{2}{c}\left(\frac{\ell_{\|}}{1-\frac{v^2}{c^2}} - \frac{\ell_\perp}{\sqrt{1-\frac{v^2}{c^2}}}\right) = \frac{2}{c}\frac{\ell_{\|} - \ell_\perp\sqrt{1-\frac{v^2}{c^2}}}{1-\frac{v^2}{c^2}}
\end{aligned}
$$

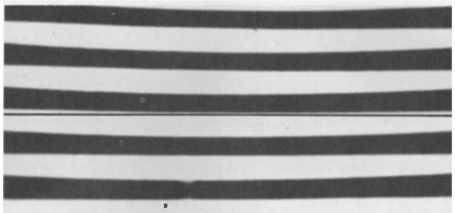
Rotation by 90^0 within few minutes:

$$
\begin{aligned}
\Delta t'(\ell_{\|}, \ell_\perp) &= -\Delta t(\ell_\perp, \ell_{\|}) = \frac{2}{c}\frac{\ell_{\|}\sqrt{1-\frac{v^2}{c^2}} - \ell_\perp}{1-\frac{v^2}{c^2}} \\
\Delta t(\ell_{\|}, \ell_\perp) - \Delta t'(\ell_{\|}, \ell_\perp) &= \frac{2}{c}\frac{(\ell_{\|} + \ell_\perp)(1 - \sqrt{1-\frac{v^2}{c^2}})}{1-\frac{v^2}{c^2}}
\end{aligned}
$$

$$\sqrt{1+\epsilon} \sim 1+\frac{\epsilon}{2} \quad \to \quad \approx \frac{2(\ell_{\parallel}+\ell_{\perp})}{c}\frac{\frac{v^2}{2c^2}}{1-\frac{v^2}{c^2}} \approx (\ell_{\parallel}+\ell_{\perp})\frac{v^2}{c^3}$$

The mirrors are not exactly perpendicular $\implies$ interference rings

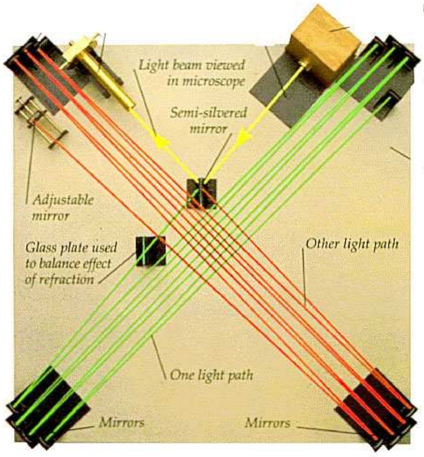
Number of shifted lines: ΔN

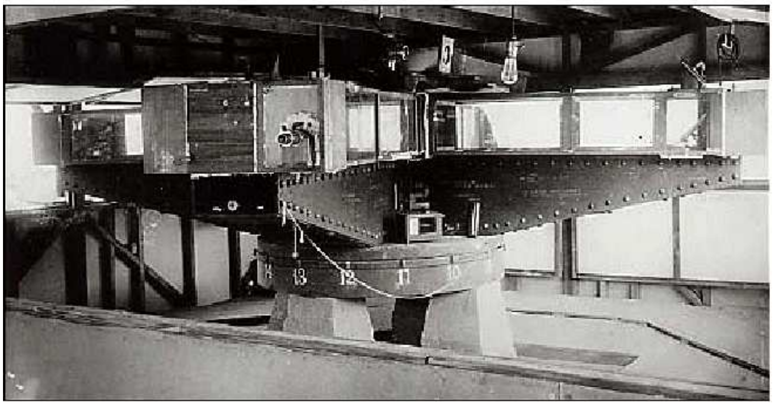


$$\Delta\ell = \lambda\Delta N = (\Delta t - \Delta t')c$$
$$\Delta N = (\Delta t - \Delta t')\frac{c}{\lambda} = \frac{v^2}{c^2}\frac{\ell_{\perp}+\ell_{\parallel}}{\lambda}$$

Assuming $v_{Earth} = 30km/s$, $\frac{v}{c} \approx 10^{-4}$ and
using $\lambda = 6\times 10^{-7}m$, $\ell = 1.2m \implies \Delta N \approx 0.04$ was not found

Michelson-Morley (1887): $\ell \to 10\ell$, $\Delta N \to 0.4$, $\Delta N_{obs} = 0 \pm 0.005$





Fitzgerald Lorentz (1892): a mysterious contraction of solids in motion:

$$\ell \to \ell\sqrt{1-\frac{v^2}{c^2}},\ \Delta t = 0 \text{ for } \ell_{\perp} = \ell_{\parallel}$$

Einstein (1905): contraction follows form the way one measures the length

Kennedy Thorndike (1932): null result with $\ell_{\perp} \neq \ell_{\parallel}$

E. The problem and its solution

The light is of the nature particle or wave?

Particle: $v_{part} = v_{source} + v_{emission}$, but $c \neq v_{source} + c$?

Wave: v_{onde} est fixed, but Michelson-Morley ?

The addition of the velocity do not apply to light?

One of the following seemingly natural assumptions is wrong:

1. The physical laws assume the same form in each reference frame.
2. The time is absolute.

Einstein (1905): No justification of point 2.

Special relativity: The physical laws assume the same form in each reference frame.

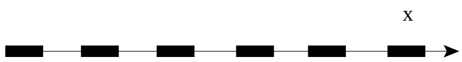
Results:

1. The speed of light is fixed by Maxwell's equations.
2. The coordinate and the velocity can be given relative to some reference object.
3. The acceleration and the derivatives $\frac{d^n \boldsymbol{x}(t)}{dt^n}$, $n \geq 2$ are absolute.

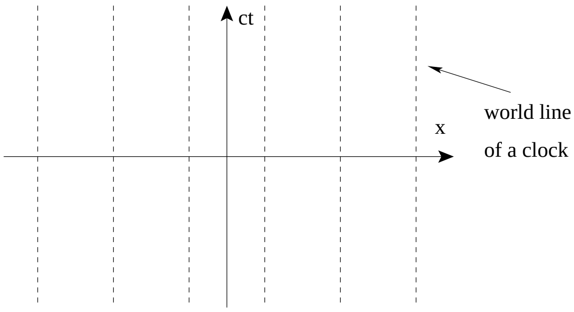
General relativity: The coordinates, together with all derivatives, $\frac{d^n \boldsymbol{x}(t)}{dt^n}$, are relative.

A more careful analysis of the measurement of the coordinates and the time is needed.

Distance: with a reference meter rod (macroscopic!)

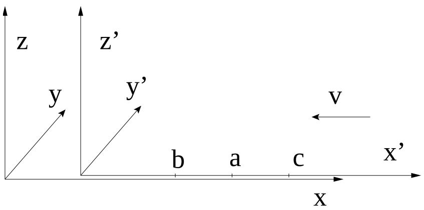


Time: synchronized standard clock at each space point



Simultaneity is relative:

a light signal $b \leftarrow a \rightarrow c$



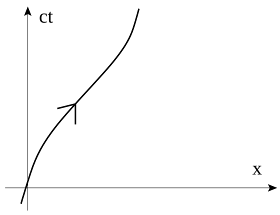
III. SPACE-TIME

A. World line

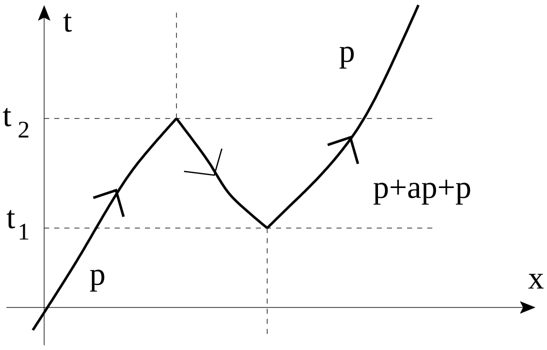
Non-relativistic motion: trajectory: $\boldsymbol{x}(t)$

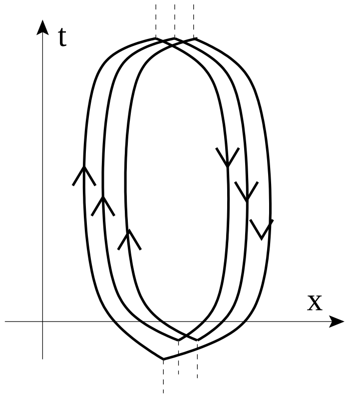
Relativistic motion: world line: $x^\mu(s) = (ct(s), \boldsymbol{x}(s))$, $\mu = 0, 1, 2, 3$.

Non-relativistic motion:



New possibility: world line $\Longrightarrow$ trajectory





length s: ordering the events

Anti-particle: $t \to -t \;\leftrightarrow\; E \to -E$

Classical mechanics: $\frac{dp}{dt} = -\frac{\partial H(p,q)}{\partial q}$, $\frac{dq}{dt} = \frac{\partial H(p,q)}{\partial p}$

Universe of a single fermion

Quantum mechanics: $i\hbar\partial_t\psi = H\psi$

Quantum field theory: $E \geq 0$

Giving up classical E.O.M.:

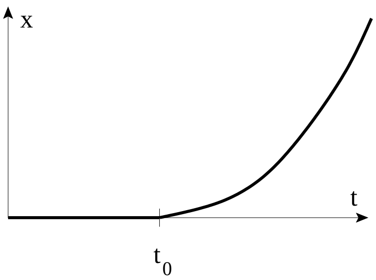
- Sufficent condition for locally unique solution:

$$\dot{x}(t) = f(x,t), \quad x(t_i) = x_i$$

  is the continuity of $\partial_x f(x,t)$

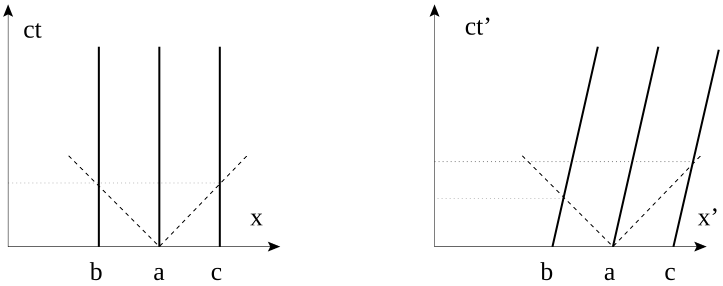
- Example:

$$\begin{aligned}
\dot{x} &= g|x|^p, \quad 0 < p < 1 \\
x(t) &= \alpha t^\beta, \quad \alpha\beta t^{\beta-1} = g(\alpha t^\beta)^p \\
\beta - 1 &= \beta p, \quad \beta = \frac{p}{1-p} \\
\alpha\beta &= g\alpha^p, \quad \frac{p\alpha}{1-p} = g\alpha^p, \quad \frac{p}{g(1-p)} = \alpha^{p-1} \\
x(t) &= \begin{cases} 0 & t < t_0, \\ \left[g\left(\frac{1}{p} - 1\right)(t - t_0)\right]^{\frac{1}{1-p}} & t \geq t_0, \end{cases}
\end{aligned}$$

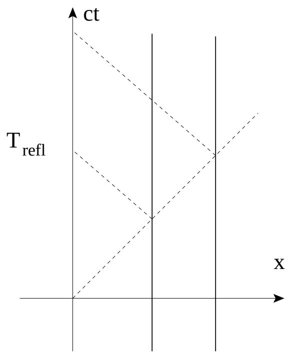


t_0 is not determined by the initial condition, pair creation is quantum phenomenon

Relative simultaneity:



Clock synchronization:

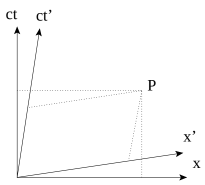


1. reference clock at $\boldsymbol{x} = 0$
2. light signal from $(ct, \boldsymbol{x}) = 0$
3. time at reflection: $t(\boldsymbol{x})$
4. time of return to $\boldsymbol{x} = 0$: $T(\boldsymbol{x})$
5. $\Delta t = \frac{T(\boldsymbol{x})}{2} - t(\boldsymbol{x})$

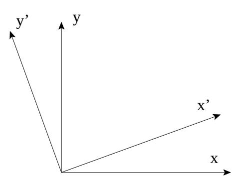
B. Lorentz transformation

Transformation between two reference frame: $(ct, \boldsymbol{x}) \to (ct', \boldsymbol{x}')$

Non-orthogonal axes

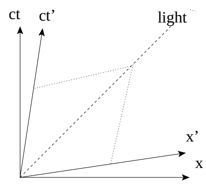


Euclidean rotation



No fixed line

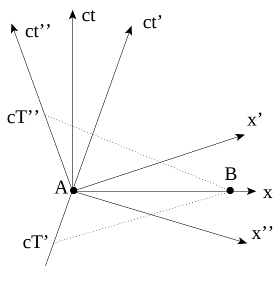
Lorentz Transformation



Light: a fixed line

Simultaneity is relative:

$T'(B) - T'(A) < T(B) - T(A) = 0 < T''(B) - T''(A)$



$$
\begin{aligned}
\text{General form:} \quad x' &= ax - bct \\
\text{boost:} \quad &= a(x - vt) \\
\text{inverse } (v \to -v)\text{:} \quad x &= a(x' + vt')
\end{aligned}
$$

Applied for the propagation of the light $x = ct$, $x' = ct'$

$$ct' = a(c-v)t,\ ct = a(c+v)t' \implies ct = a(c+v)\frac{a}{c}(c-v)t,\ a^2 = \frac{c^2}{c^2-v^2},\ a = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\begin{aligned} x' &= \frac{x-vt}{\sqrt{1-\frac{v^2}{c^2}}}, & x = \frac{x'+vt'}{\sqrt{1-\frac{v^2}{c^2}}} \\ t' &= \frac{t-\frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} (\leftarrow\ x = ct), & t = \frac{t'+\frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \end{aligned}$$

$\frac{v}{c} \to 0$: Galilean transformations

No change in the orthogonal directions: $\boldsymbol{v} = (v, 0, 0)$, $y = y'$, $z = z'$

Lorentz (1904) : compensation

Einstein (1905): way of observation

Lorentz symmetry is obeyed by all interactions

C. Addition of the velocity

Two different reference frames: S and S': $x_\parallel = \frac{x'_\parallel + ut'}{\sqrt{1-\frac{u^2}{c^2}}}$, $\quad \boldsymbol{x}_\perp = \boldsymbol{x}'_\perp$, $\quad t = \frac{t' + \frac{ux'_\parallel}{c^2}}{\sqrt{1-\frac{u^2}{c^2}}}$

$$\begin{aligned} \boldsymbol{v}' &= \frac{d\boldsymbol{x}'}{dt'} \ \to\ \boldsymbol{v} = \frac{d\boldsymbol{x}}{dt} \\ \Delta t \ \to\ \Delta x_\parallel &= \frac{(v'_\parallel + u)\Delta t'}{\sqrt{1-\frac{u^2}{c^2}}}, \qquad \Delta\boldsymbol{x}_\perp = \boldsymbol{v}'_\perp \Delta t', \qquad \Delta t = \frac{(1+\frac{uv'_\parallel}{c^2})\Delta t'}{\sqrt{1-\frac{u^2}{c^2}}} \\ v_\parallel &= \frac{\Delta x_\parallel}{\Delta t} = \frac{v'_\parallel + u}{1+\frac{uv'_\parallel}{c^2}} = \begin{cases} v'_\parallel + u & u, v'_\parallel \ll c \\ c & v'_\parallel \ll c, u \approx c, \ \text{ou} \ \ u \ll c, v'_\parallel \approx c \end{cases} \\ \boldsymbol{v}_\perp &= \boldsymbol{v}'_\perp \frac{\sqrt{1-\frac{u^2}{c^2}}}{1+\frac{uv'_x}{c^2}} = \begin{cases} \boldsymbol{v}'_\perp & u \ll c \\ 0 & u \approx c \end{cases} \end{aligned}$$

D. Invariant distance

Euclidean geometry: the invariance of $s^2 = (\boldsymbol{x}_2 - \boldsymbol{x}_1)^2$

identifies the symmetries (translations + rotations)

Minkowski geometry: the invariance of $s^2 = c^2(t_2 - t_1)^2 - (\boldsymbol{x}_2 - \boldsymbol{x}_1)^2$

identifies the symmetries (translations + Lorentz transformations)

Proof A: (by Lorentz transformation)

$$x' = \frac{x - vt}{\sqrt{1-\frac{v^2}{c^2}}}, \quad t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$s^2 = c^2t^2 - x^2 \to \frac{(ct - \frac{vx}{c})^2 - (x-vt)^2}{1-\frac{v^2}{c^2}} = \frac{(c^2t^2 - x^2)(1-\frac{v^2}{c^2})}{1-\frac{v^2}{c^2}} = c^2t^2 - x^2$$

Proof B: (by consistency)

- $s^2 = 0$ the possibility of exchanging light signal is Lorentz invariant

- $s^2 \neq 0$? Three reference frames, S_0 and S_j, $\boldsymbol{v}_{S_0 \to S_j} = \boldsymbol{v}_j$, $j = 1, 2$, $|\boldsymbol{v}_1|, |\boldsymbol{v}_2| \ll c$

Step 1. $s_j^2 = F(|\boldsymbol{v}_j|, s_0^2)$

Step 2. $F(|\boldsymbol{v}|, 0) = 0$

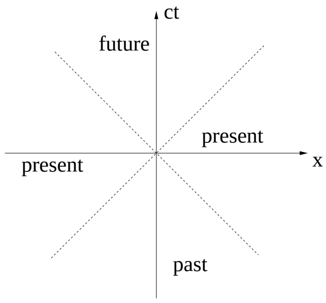
Step 3. continuity of $F(v, s^2)$ in s^2 at $s^2 = 0$: $ds_j^2 = F(|\boldsymbol{v}_j|, ds_0^2) \approx \underbrace{\frac{\partial F(|\boldsymbol{v}_j|, s^2)}{\partial s^2}_{|s^2=0}}_{a(|\boldsymbol{v}_j|)} ds_0^2$

Step 4. $ds_j^2 = a(|\boldsymbol{v}_j|)ds_0^2, \quad ds_2^2 = a(|\boldsymbol{v}_1 - \boldsymbol{v}_2|)ds_1^2, \quad a(|\boldsymbol{v}_1 - \boldsymbol{v}_2|) = \frac{a(|\boldsymbol{v}_2|)}{a(|\boldsymbol{v}_1|)} \to a = 1$

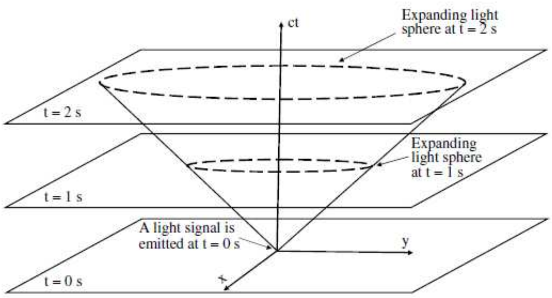
E. Minkowski geometry

Three different kinds of space-time intervals:

- time-like: $c^2(t_1 - t_2)^2 - (\boldsymbol{x}_1 - \boldsymbol{x}_2)^2 > 0$
- space-like: $c^2(t_1 - t_2)^2 - (\boldsymbol{x}_1 - \boldsymbol{x}_2)^2 < 0$
- light-like: $c^2(t_1 - t_2)^2 - (\boldsymbol{x}_1 - \boldsymbol{x}_2)^2 = 0$



past, present, future



Light cone

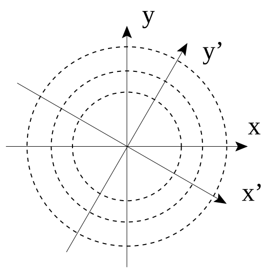
Change of scale:

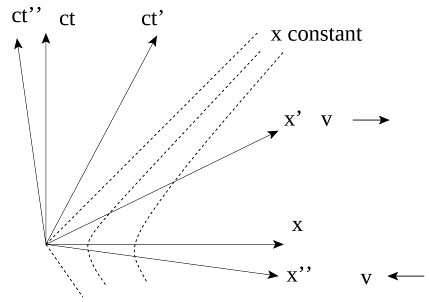
Euclidean:

$x^2 + y^2 = R^2$

Minkowski:

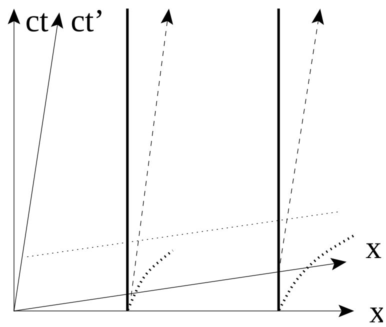
$(ct)^2 - x^2 = s^2$





IV. PHYSICAL PHENOMENAS

A. Lorentz contraction of the length

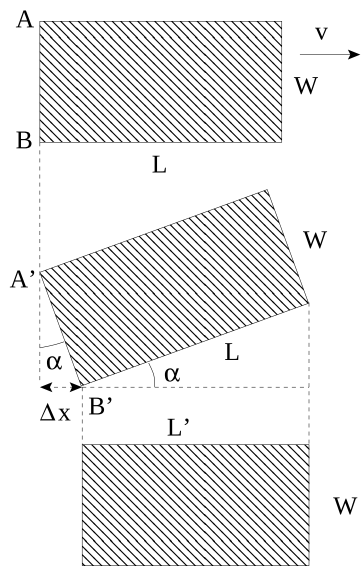


$\ell = x_2 - x_1$

$\ell' = x'_2 - x'_1$ **simultaneous** coordinates

$$\begin{aligned} x_a &= \frac{x'_a + vt'}{\sqrt{1-\frac{v^2}{c^2}}} \\ \ell &= \frac{x'_2 - x'_1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{\ell'}{\sqrt{1-\frac{v^2}{c^2}}} \\ \ell' &= \ell\sqrt{1-\frac{v^2}{c^2}} \end{aligned}$$

A simple model: apparent rotation of a rectangular board

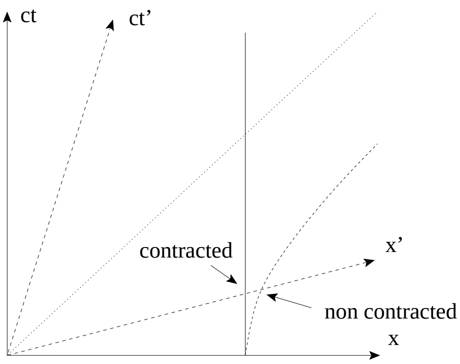


The light, propagating from A_0 and B_0 arrive

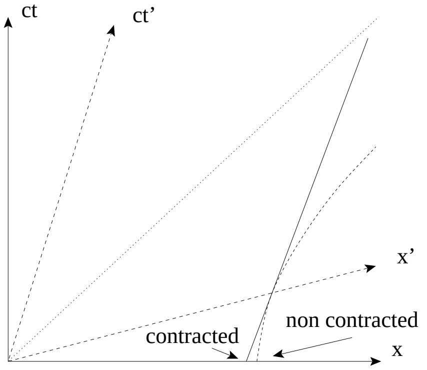
at the **same time**:

$$\begin{aligned} \Delta t &= \frac{W}{c} \\ \Delta x &= v\Delta t = W\frac{v}{c} \\ \sin\Theta &= \frac{v}{c}, \quad \cos\Theta = \sqrt{1-\frac{v^2}{c^2}} \\ L' &= L\sqrt{1-\frac{v^2}{c^2}} \leftarrow \text{Lorentz contraction} \end{aligned}$$

The contraction is relative:



Rod at rest



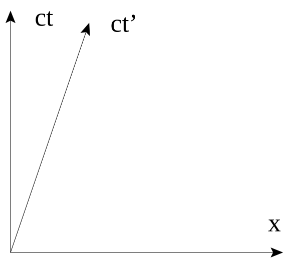
Moving rod

B. Time dilatation

Moving clock: Proper time: $t_0 = \frac{\sqrt{s^2}}{c}$

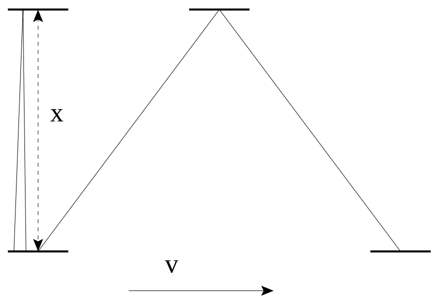
$$\begin{aligned} s^2 &= c^2 t_0^2 = c^2 t^2 - x^2 = c^2 t^2 \left(1 - \frac{v^2}{c^2}\right) \\ t &= \frac{t_0}{\sqrt{1 - \frac{v^2}{c^2}}} \end{aligned}$$

$t > t_0$: slowing down



Optical clock:

$$\begin{aligned} c^2 t_0^2 &= x^2 \\ c^2 t'^2 &= t'^2 v^2 + t_0'^2 c^2 \\ t' &= \frac{t_0}{\sqrt{1 - \frac{v^2}{c^2}}} \end{aligned}$$

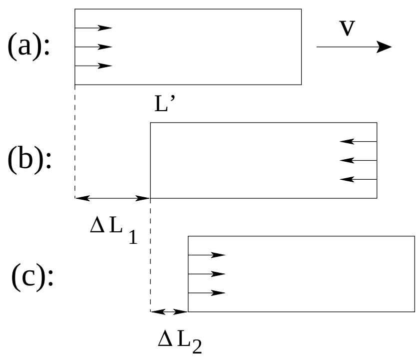


Time dilatation $\implies$ Lorentz contraction

Moving rod with mirrors:

$$\begin{aligned} c\Delta t_1 &= \ell + v\Delta t_1 \\ c\Delta t_2 &= \ell - v\Delta t_2 \\ 2t' &= \Delta t_1 + \Delta t_2 \\ &= \frac{\ell'}{c-v} + \frac{\ell'}{c+v} = \frac{2\ell' c}{c^2 - v^2} \\ t' &= \frac{\ell'}{c} \frac{1}{1 - \frac{v^2}{c^2}} \end{aligned}$$

$$\ell' = \ell_0 \sqrt{1 - \frac{v^2}{c^2}} \leftrightarrow t' = \frac{t_0}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \text{for} \quad \ell_0 = t_0 c$$

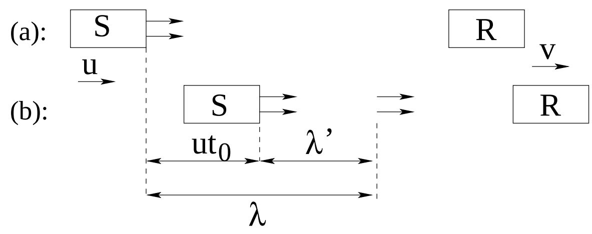


C. Doppler effect

Non-relativistic: sound velocity: w

Following one cycle:

$$\begin{aligned} w\Delta t &= \lambda' + u\Delta t \\ \lambda' &= (w-u)\Delta t = \frac{w-u}{\nu} = \frac{w-v}{\nu'} \\ \nu' &= \nu\frac{w-v}{w-u} \end{aligned}$$

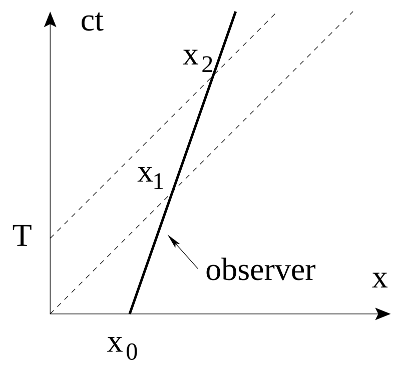


Stationary source: $u = 0$, $\nu' = \nu\left(1 - \frac{v}{w}\right)$, $\nu' = 0$ for $v = w$

Stationary receiver: $v = 0$, $\nu' = \frac{\nu}{1-\frac{u}{w}}$, $\nu' = \infty$ for $u = w$

Relativistic (light): Two pulses with time difference T, speed of the detector is v

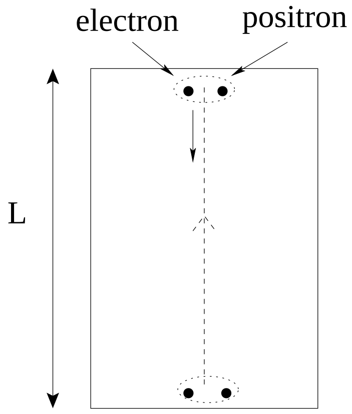
$$\begin{aligned} x_1 &= ct_1 = x_0 + vt_1 & &\rightarrow\ t_1 = \frac{x_0}{c-v} \\ x_2 &= c(t_2 - T) = x_0 + vt_2 & &\rightarrow\ t_2 = \frac{x_0 + cT}{c-v} \\ t_2 - t_1 &= \frac{T}{1-\frac{v}{c}}, & &x_2 - x_1 = \frac{vT}{1-\frac{v}{c}} \end{aligned}$$



$$\begin{aligned} T' = t_2' - t_1' &= \frac{t_2 - t_1 - \frac{v}{c^2}(x_2 - x_1)}{\sqrt{1-\frac{v^2}{c^2}}} \\ &= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(\frac{T}{1-\frac{v}{c}} - \frac{v}{c^2}\frac{vT}{1-\frac{v}{c}}\right) \\ &= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\frac{T}{1-\frac{v}{c}}\left(1-\frac{v^2}{c^2}\right) = T\frac{\sqrt{1-\frac{v^2}{c^2}}}{1-\frac{v}{c}} = T\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} \\ \nu' &= \nu\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} \neq \underbrace{\nu\left(1-\frac{v}{c}\right)}_{\text{non-relativistic}} \end{aligned}$$

Static gravitational field:

$$\begin{aligned}
E_\downarrow(e^-e^+) &= \underbrace{E_\uparrow(e^-e^+)}_{2mc^2} + 2m\Delta U \\
&= E_\uparrow(e^-e^+)\left(1+\frac{\Delta U}{c^2}\right) \\
E_\uparrow(e^-e^+) &= E_\uparrow(\gamma), \quad E_\downarrow(e^-e^+) = E_\downarrow(\gamma) \\
E &= \hbar\omega \\
\frac{\omega_\downarrow}{\omega_\uparrow} &= \frac{E_\downarrow(e^-e^+)}{E_\uparrow(e^-e^+)} = 1 + \underbrace{\frac{\Delta U}{c^2}}_{z} \\
z &= \frac{\Delta U}{c^2}
\end{aligned}$$



Time-dependent gravitational field:

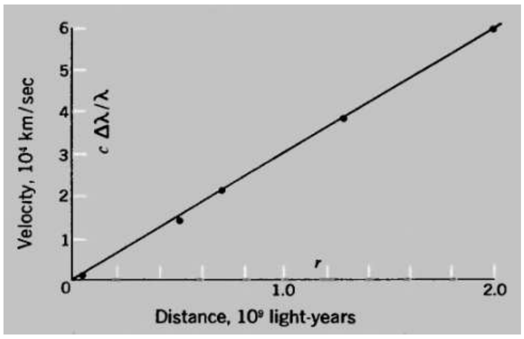
Robertson-Walker model of the Universe

Universe in expansion: $\lambda_{em} < \lambda_{obs}$

Red shift: $\dfrac{\omega_{em}}{\omega_{obs}} = \dfrac{\lambda_{obs}}{\lambda_{em}} = 1 + z > 1$

Hubble constant: $z = \dfrac{H}{c}\ell$

age of the Universe: $T_U = \dfrac{1}{H}$, $z = \dfrac{\frac{\ell}{c}}{T_U}$

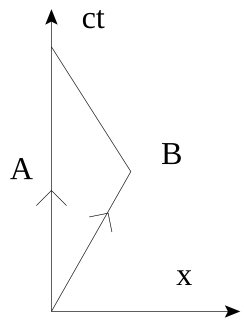


D. Paradoxes

Twins:

A: rests, B: leaves and returns

Which one is older when they meet again?



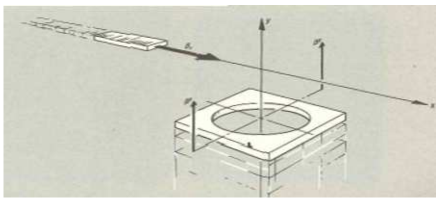
A rod and a circle:

$\ell = 2r$, $\boldsymbol{v}_b = (u,0,0)$, $\boldsymbol{v}_c = (0,v,0)$

$t = 0$: the center of the rod and

the origin of the circle coincide

Can they cross each other?



A spear and a stable:

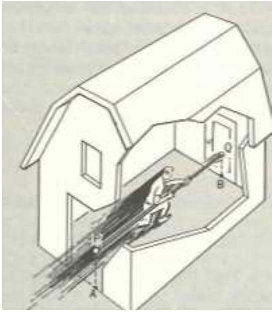
$\ell_{spear} = 20m$, $\ell_{stable} = 10m$

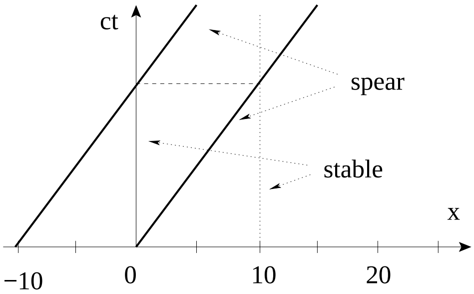
$\sqrt{1-\frac{v^2}{c^2}} = \frac{1}{2}$

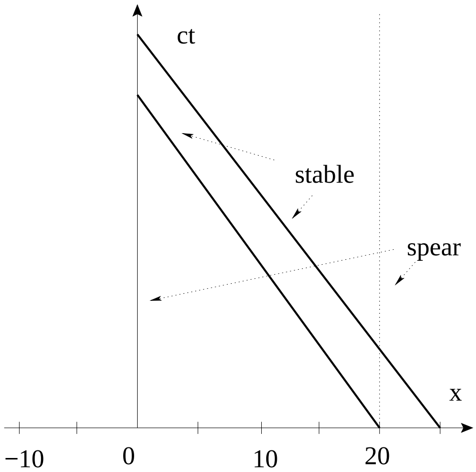
Reference frame of the stable: $\ell'_{spear} = 10m$

Reference frame of the spear: $\ell'_{stable} = 5m$

Can the spear enter into the stable?







V. RELATIVISTIC MECHANICS

A. Vectors and tensors

The components of the four-vector, $x^\mu = (ct, \boldsymbol{x}) = (x^0, \boldsymbol{x})$, $\mu = 0, 1, 2, 3$,

are mixed by Lorentz transformations

Invariant length: to characterize the Lorentz transformations

$$s^2 = x^{02} - \boldsymbol{x}^2 = x^\mu g_{\mu\nu} x^\nu = xx$$

Metric tensor: to define the scalar product

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Scalar product:

$$xy = \sum_{\mu\nu} x^\mu g_{\mu\nu} y^\nu$$

To suppress g one uses two representation for each vector u:

- a covariant, u_μ, and
- a contravariant, u^μ:

$$\begin{aligned} u_\mu &= g_{\mu\nu}u^\nu, \quad u^\mu = g^{\mu\nu}u_\nu \quad \to \quad xy = x^\mu y_\mu = x_\mu y^\mu \\ u_\mu &= g_{\mu\nu}g^{\nu\rho}u_\rho \quad \to \quad g_{\mu\nu}g^{\nu\rho} = g_{\mu\cdot}^{\cdot\rho} = \delta_\mu^\rho \end{aligned}$$

Einstein convention:

$$\sum_\mu (\cdots)^\mu(\cdots)_\mu \to (\cdots)^\mu(\cdots)_\mu$$

Lorentz group: Family of linear transformations

$$x^\mu \to x'^\mu = \Lambda^{\mu}_{\cdot\,\nu}{}^{\cdot} x^\nu,$$

preserving the invariant length,

$$xy = x^{\mu'}\Lambda^{\mu}_{\cdot\,\mu'}{}^{\cdot} g_{\mu\nu}\Lambda^{\nu}_{\cdot\,\nu'}{}^{\cdot} y^{\nu'}, \qquad g_{\mu\nu} = \Lambda^{\mu'}_{\cdot\,\mu}{}^{\cdot} g_{\mu'\nu'}\Lambda^{\nu'}_{\cdot\,\nu}{}^{\cdot}.$$

It has 6 dimensions, 3 for spatial rotation,

$$\Lambda = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & R \end{pmatrix}, \quad R^{\mathrm{tr}}R = \mathbb{1}$$

and another 3 for boosts: $\boldsymbol{x} = \boldsymbol{x}_\parallel + \boldsymbol{x}_\perp$, $\boldsymbol{x}_\parallel\boldsymbol{x}_\perp = \boldsymbol{v}\boldsymbol{x}_\perp = 0$.

$$\boldsymbol{x}' = \alpha(\boldsymbol{x}_\parallel - \boldsymbol{v}t) + \gamma\boldsymbol{x}_\perp, \qquad t' = \beta\left(t - \frac{\boldsymbol{x}_\parallel\boldsymbol{v}}{\tilde{c}^2}\right)$$

Invariance:

$$\begin{aligned} c^2t^2 - \boldsymbol{x}^2 &= c^2\beta^2\left(t - \frac{\boldsymbol{x}_\parallel\boldsymbol{v}}{\tilde{c}^2}\right)^2 - \alpha^2(\boldsymbol{x}_\parallel - \boldsymbol{v}t)^2 - \gamma\boldsymbol{x}_\perp^2 \\ &= c^2t^2\left(\beta^2 - \alpha^2\frac{\boldsymbol{v}^2}{c^2}\right) - \boldsymbol{x}_\parallel^2\left(\alpha^2 - \frac{c^2}{\tilde{c}^2}\frac{\boldsymbol{v}^2}{\tilde{c}^2}\beta^2\right) + 2\boldsymbol{v}\boldsymbol{x}_\parallel t\left(\alpha^2 - \frac{c^2}{\tilde{c}^2}\beta^2\right) \\ \mathcal{O}\left(\boldsymbol{x}_\perp^2\right): \ \gamma &= \pm 1 \to 1 \quad \leftarrow \quad v = 0 \\ \mathcal{O}\left(\boldsymbol{x}_\perp\boldsymbol{v}\right): \ 0 &= \alpha^2 - \frac{c^2}{\tilde{c}^2}\beta^2 \\ \mathcal{O}\left(\boldsymbol{x}_\parallel^2\right): \ -1 &= \frac{c^2}{\tilde{c}^2}\frac{\boldsymbol{v}^2}{\tilde{c}^2}\beta^2 - \alpha^2 \ \to \ \beta^2 = \frac{\tilde{c}^2}{c^2}\frac{1}{1 - \frac{\boldsymbol{v}^2}{\tilde{c}^2}} \\ \mathcal{O}\left(c^2t^2\right): \ 1 &= \beta^2 - \alpha^2\frac{\boldsymbol{v}^2}{c^2} = \frac{\tilde{c}^2}{c^2}\frac{1}{1 - \frac{c^2}{\tilde{c}^2}\frac{\boldsymbol{v}^2}{c^2}}\left(1 - \frac{c^2}{\tilde{c}^2}\frac{\boldsymbol{v}^2}{c^2}\right) \ \to \ \tilde{c}^2 = c^2 \\ \alpha &= \beta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \leftarrow \quad v = 0 \end{aligned}$$

$$\boldsymbol{x}'_\parallel = \frac{\boldsymbol{x}_\parallel - \boldsymbol{v}t}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad ct' = \frac{ct - \frac{\boldsymbol{v}\boldsymbol{x}_\parallel}{c}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Inverse: $\boldsymbol{v} \to -\boldsymbol{v}$

$$\boldsymbol{x}_\parallel = \frac{\boldsymbol{x}'_\parallel + \boldsymbol{v}t'}{\sqrt{1-\frac{v^2}{c^2}}}, \quad ct = \frac{ct' + \frac{\boldsymbol{v}\boldsymbol{x}'_\parallel}{c}}{\sqrt{1-\frac{v^2}{c^2}}}$$

Exercise: internal consistency, $\Lambda = \Lambda^{\mu}_{\cdot\ \nu}$, $g = g_{\mu\nu}$, $g^{\rm tr} = g$,

$$\begin{aligned}
g &= \Lambda^{\rm tr} g \Lambda, \quad \leftrightarrow \quad g_{\mu\nu} = \Lambda^{\mu'}_{\cdot\ \mu} g_{\mu'\nu'} \Lambda^{\nu'}_{\cdot\ \nu} \\
\Lambda^{-1} &= g^{-1}\Lambda^{\rm tr} g = (g\Lambda g^{-1})^{\rm tr}, \quad \leftrightarrow \quad \Lambda^{-1\mu}_{\cdot\ \nu} = g^{\mu\mu'}\Lambda^{\nu'}_{\cdot\ \mu'} g_{\nu'\nu} = \Lambda_{\nu\cdot}^{\cdot\ \mu} \quad \text{(orth.)} \\
x'^\mu &= (\Lambda x)^\mu = \Lambda^{\mu}_{\cdot\ \nu} x^\nu \\
x^\mu &= (g\Lambda g^{-1})_{\nu\cdot}^{\cdot\ \mu} x'^\nu = x'^\nu \Lambda_{\nu\cdot}^{\cdot\ \mu} = (x'\Lambda)^\mu \\
x'_\mu &= (g\Lambda x)_\mu = (g\Lambda g^{-1} g x)_\mu = \Lambda_{\mu\cdot}^{\cdot\ \nu} x_\nu = (\Lambda x)_\mu \\
x_\mu &= x'^\nu g_{\nu\lambda} g^{\lambda\rho} \Lambda_{\rho\mu} = x'_\lambda \Lambda^{\lambda}_{\cdot\ \mu} = (x'\Lambda)_\mu \\
\text{N.B. rotations}: \quad \boldsymbol{x}' &= R\boldsymbol{x}, \quad \boldsymbol{x} = R^{\rm tr}\boldsymbol{x}' = \boldsymbol{x}'R
\end{aligned}$$

$$\begin{aligned}
\text{contravariant tensor} &: T^{\mu_1\cdots\mu_n} = \Lambda^{\mu_1}_{\ \ \nu_1}\cdots\Lambda^{\mu_n}_{\ \ \nu_n} T^{\nu_1\cdots\nu_n} \\
\text{covariant tensor} &: T_{\mu_1\cdots\mu_n} = \Lambda_{\mu_1}^{\ \ \nu_1}\cdots\Lambda_{\mu_n}^{\ \ \nu_n} T_{\nu_1\cdots\nu_n} \\
\text{mixed tensor} &: T^{\rho_1\cdots\rho_m}_{\mu_1\cdots\mu_n} = \Lambda^{\rho_1}_{\ \ \kappa_1}\cdots\Lambda^{\rho_m}_{\ \ \kappa_m}\Lambda_{\mu_1}^{\ \ \nu_1}\cdots\Lambda_{\mu_n}^{\ \ \nu_n} T^{\kappa_1\cdots\kappa_m}_{\nu_1\cdots\nu_n}
\end{aligned}$$

Invariant tensors:

1. $g_{\mu\nu} = \Lambda^{\mu'}_{\ \ \mu} g'_{\mu'\nu'} \Lambda^{\nu'}_{\ \ \nu} \implies g_{\mu\nu},\ g^{\mu\nu},\ g^{\nu}_{\mu}$

2. $\epsilon^{0123} = 1$, $\epsilon'^{\mu\nu\rho\sigma} = \Lambda^{\mu}_{\ \ \mu'}\Lambda^{\nu}_{\ \ \nu'}\Lambda^{\rho}_{\ \ \rho'}\Lambda^{\sigma}_{\ \ \sigma'}\epsilon^{\mu'\nu'\rho'\sigma'} = \epsilon^{\mu\nu\rho\sigma}\det\Lambda \implies \epsilon^{\mu\nu\rho\sigma}$ (pseudo tensor)

B. Relativistic generalization of the Newtonian mechanics

Four-velocity: $ds = dx^0\sqrt{1-\frac{v^2}{c^2}}$

$$\begin{aligned}
ds^2 &= dx^{02} - d\boldsymbol{x}^2 = dx^{02}\left[1 - \left(\frac{d\boldsymbol{x}}{dx^0}\right)^2\right] = dx^{02}\left(1 - \frac{\dot{\boldsymbol{x}}^2}{c^2}\right) \\
ds &= dx^0\sqrt{1-\frac{v^2}{c^2}} \\
u^\mu &= \frac{dx^\mu(s)}{ds} = \dot{x}(s) = \left(\frac{dx^0}{ds}, \frac{dx^0}{ds}\frac{\boldsymbol{v}}{c}\right) = \left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}, \frac{\frac{\boldsymbol{v}}{c}}{\sqrt{1-\frac{v^2}{c^2}}}\right)
\end{aligned}$$

Four-acceleration:

$$\dot{u}^\mu = \frac{du^\mu}{ds}, \qquad u^2 = 1 \quad \longrightarrow \quad \dot{u}u = 0$$

Four-momentum:

$$p^\mu = mcu^\mu = \left(\frac{mc}{\sqrt{1-\frac{v^2}{c^2}}}, \frac{m\boldsymbol{v}}{\sqrt{1-\frac{v^2}{c^2}}}\right) = \left(\frac{E}{c}, \boldsymbol{p}\right), \qquad p^2 = m^2c^2$$

N.B. (i) $c \to \infty$, $\boldsymbol{p} \to m\boldsymbol{v}$, (ii) $v \to c$, $\boldsymbol{p} \to \infty$

Four-force:

$$\frac{dp^\mu}{ds} = \frac{d}{ds}\left(mc\frac{dx^\mu}{ds}\right) = K^\mu$$

Spatial components:

$$\begin{aligned}
\frac{dt}{ds}\frac{d}{dt}\left(mc\frac{dt}{ds}\frac{d\boldsymbol{x}}{dt}\right) &= \boldsymbol{K} \\
m(v) &= m\frac{dx^0}{ds} = \frac{m}{\sqrt{1-\frac{v^2}{c^2}}} \\
\frac{d}{dt}\left(m(v)\frac{d\boldsymbol{x}}{dt}\right) &= \frac{d}{dt}\boldsymbol{p} = \frac{ds}{dt}\boldsymbol{K} = \boldsymbol{F}
\end{aligned}$$

Temporal component: (energy equation)

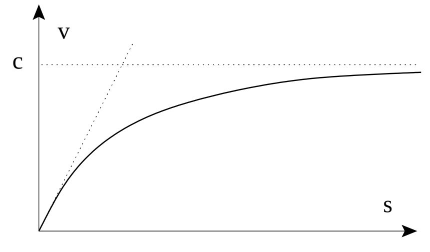
$$\begin{aligned}
\frac{d}{ds}\left(mc\frac{dx^0}{ds}\right) &= \frac{d}{ds}\frac{mc}{\sqrt{1-\frac{v^2}{c^2}}} = K^0 \\
Ku &= 0 \qquad \longrightarrow \qquad \underbrace{K^0}_{\frac{dt}{ds}\frac{d}{dt}cm(v)}\frac{dx^0}{ds} = \underbrace{\boldsymbol{K}}_{\frac{dt}{ds}\boldsymbol{F}}\underbrace{\boldsymbol{u}}_{\frac{dt}{ds}\boldsymbol{v}} \\
\frac{dm(v)c^2}{dt} &= \boldsymbol{F}\boldsymbol{v} \qquad \longrightarrow \qquad m(v)c^2 = \int \boldsymbol{F}\boldsymbol{v}dt = \int \boldsymbol{F}d\boldsymbol{r} \\
\text{let} \quad \boldsymbol{F} &= -\boldsymbol{\nabla}\phi \qquad \longrightarrow \qquad E = \underbrace{m(v)c^2}_{cp^0} + \phi \quad \text{is constant}
\end{aligned}$$

Energy:

$$\begin{aligned}
p^\mu &= \left(\frac{E}{c}, m(v)\boldsymbol{v}\right), \qquad \frac{E^2}{c^2} = \boldsymbol{p}^2 + m^2c^2, \qquad E(\boldsymbol{p}) = \pm c\sqrt{\boldsymbol{p}^2 + m^2c^2} \\
\frac{\partial|E(\boldsymbol{p})|}{\partial\boldsymbol{p}} &= \frac{\boldsymbol{p}c}{\sqrt{\boldsymbol{p}^2+m^2c^2}} = \frac{\frac{\boldsymbol{p}}{m}}{\sqrt{1+\frac{\boldsymbol{p}^2}{m^2c^2}}} = \frac{\frac{\boldsymbol{v}}{\sqrt{1-\frac{v^2}{c^2}}}}{\sqrt{1+\frac{\boldsymbol{v}^2}{c^2(1-\frac{\boldsymbol{v}^2}{c^2})}}} = \boldsymbol{v}, \quad |\boldsymbol{v}| \le c
\end{aligned}$$

Ex. 1: Constant three-force, $\frac{d}{dt}\frac{mv}{\sqrt{1-\frac{v^2}{c^2}}} = F$, $x_i = v_i = 0$ (non Lorentz inv.!)

$$\begin{aligned}
Ft &= \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}}, \quad 1 - \frac{v^2}{c^2} = \left(\frac{mv}{Ft}\right)^2 \\
c^2 - v^2 &= v^2\left(\frac{mc}{Ft}\right)^2 \ \to \ c^2 = v^2\left[1 + \left(\frac{mc}{Ft}\right)^2\right] \\
v &= \frac{c}{\sqrt{1+\left(\frac{mc}{Ft}\right)^2}} \approx \begin{cases} t\frac{F}{m} & Ft \ll mc \\ c & Ft \gg mc \end{cases}
\end{aligned}$$

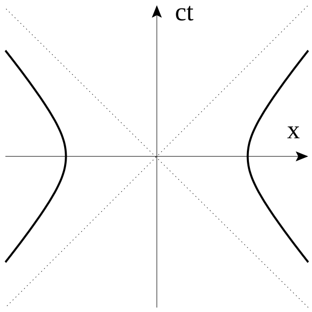


Ex. 2: Constant force, $\frac{dp}{ds} = K$

Co-moving reference frame, $x_i = v_i = 0$: $u = \dot{x} = (1, \mathbf{0})$, $\dot{u} = (0, \frac{\boldsymbol{a}}{c^2})$, $\dot{u}^2 = -\frac{\boldsymbol{a}^2}{c^4}$

$$u^\mu = \left(\cosh\frac{a}{c^2}s, \sinh\frac{a}{c^2}s\right), \quad \dot{u}^\mu = \frac{a}{c^2}\left(\sinh\frac{a}{c^2}s, \cosh\frac{a}{c^2}s\right)$$

$$x^\mu = \frac{c^2}{a}\left(\sinh\frac{a}{c^2}s, \cosh\frac{a}{c^2}s - 1\right)$$



C. Interactions

$$\begin{aligned} E.M. \quad \ddot{x}^\mu_a &= F^\mu_a(x_1, \ldots, x_n), \quad a = 1, \ldots, n \\ C.I. \quad x_a(s_i) &= x_{ia}, \ \dot{x}_a(s_i) = u_a \end{aligned}$$

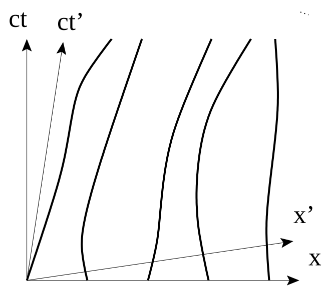
Problems:

1. The initial conditions are imposed

   on the spatial hyper-surface, $t = t_i$

   and the Lorentz boost mixes into the dynamics:

$$\dot{x}\ddot{x} = 0 \quad \to \quad u_a F_a = 0$$



2. The four-force, $F^\mu_a(x_1, \ldots, x_n)$, represents a superluminal effect

No-Go theorem: there is no relativistic interaction, mediated by a force of the type

$$F^\mu_a(x_1, \ldots, x_n) \neq 0$$

e.g.

$$F^\mu_a(x_1, \ldots, x_n) = \sum_{b \neq a}(x^\mu_a - x^\mu_b) f((x_a - x_b)^2) \quad \to \quad (x_a - x_b)\dot{x}_a \neq 0$$

Solution: - distribute degrees of freedom at each space point

- transfer the excitations with a limited speed

$\Longrightarrow$ the field, $\phi(t, \boldsymbol{x})$, appears in physics

VI. VARIATIONAL FORMALISM OF A POINT PARTICLE

Need of an equation of motion which is independent of the choice of the coordinate system.

A. A point on a line

Goal: an equation to identify $x_{cl} \in \mathbb{R}$: use a function with extreme at c_{cl} only

$$S(x) \quad \to \quad \frac{dS(x)}{dx}_{|x=x_{cl}} = 0$$

Reparametrization of the line (change of coordinate system): $x \to y$

$$\frac{dS(x(y))}{dy}_{|y=y_{cl}} = \underbrace{\frac{dS(x)}{dx}_{|x=x_{cl}}}_{0} \frac{dx(y)}{dy}_{|y=y_{cl}} = 0$$

Variational principle, an alternative definition: $x \to x + \delta x$

$$\begin{aligned} S(x_{cl} + \delta x) &= S(x_{cl}) + \delta S(x_{cl}) \\ &= S(x_{cl}) + \delta x \underbrace{S'(x_{cl})}_{0} + \frac{\delta x^2}{2} S''(x_{cl}) + \mathcal{O}\left(\delta x^3\right) \\ \Longrightarrow \quad \delta S(x_{cl}) &= \mathcal{O}\left(\delta x^2\right) \qquad \text{(manifestly coordinate independent)} \end{aligned}$$

B. Non-relativistic particle

$x_{cl}(t_i) = x_i$, $x_{cl}(t_f) = x_f \quad \Longrightarrow \quad x_{cl}(t)$

Variational principle: $x(t) \to x(t) + \delta x(t)$, $\delta x(t_i) = \delta x(t_f) = 0$

$$\begin{aligned} S[x(\cdot)] &= \int_{t_i}^{t_f} dt L(x(t), \dot{x}(t)) \\ \delta S[x(\cdot)] &= \int_{t_i}^{t_f} dt L\left(x(t) + \delta x(t), \dot{x}(t) + \frac{d}{dt}\delta x(t)\right) - \int_{t_i}^{t_f} dt L(x(t), \dot{x}(t)) \\ &= \int_{t_i}^{t_f} dt \Bigg[L(x(t), \dot{x}(t)) + \delta x(t) \frac{\delta L(x(t), \dot{x}(t))}{\delta x} \\ &\quad + \frac{d}{dt}\delta x(t) \frac{\delta L(x(t), \dot{x}(t))}{\delta \dot{x}} + \mathcal{O}\left(\delta x(t)^2\right) - \int_{t_i}^{t_f} dt L(x(t), \dot{x}(t)) \Bigg] \\ &= \int_{t_i}^{t_f} dt \delta x(t) \left[\frac{\delta L(x(t), \dot{x}(t))}{\delta x} - \frac{d}{dt} \frac{\delta L(x(t), \dot{x}(t))}{\delta \dot{x}} \right] \\ &\quad + \underbrace{\delta x(t)}_{0} \frac{\delta L(x(t), \dot{x}(t))}{\delta \dot{x}} \Bigg|_{t_f}^{t_i} + \mathcal{O}\left(\delta x(t)^2\right) \end{aligned}$$

Euler-Lagrange equation: $0 = \frac{\delta L(x, \dot{x})}{\delta x} - \frac{d}{dt} \frac{\delta L(x, \dot{x})}{\delta \dot{x}}$

Lagrangian: $L = T - U = \frac{m}{2}\dot{\boldsymbol{x}}^2 - U(\boldsymbol{x}) \quad \Longrightarrow \quad m\ddot{\boldsymbol{x}} = -\boldsymbol{\nabla} U(\boldsymbol{x})$

Generalized momentum: $p = \frac{\delta L}{\delta \dot{x}}$, E-L equation: $\dot{p} = \frac{\delta L}{\delta x}$

Cyclic coordinate: $\frac{\delta L}{\delta x_{cycl}} = 0$, p_{cycl} conserved

Energy: Legendre transf., $H(p, x) = p\dot{x} - L(x, \dot{x}) = T + U$, $\dot{H} = \dot{p}\dot{x} + p\ddot{x} - \dot{x}\frac{\delta L}{\delta x} - \ddot{x}\frac{\delta L}{\delta \dot{x}} = 0$

Examples:

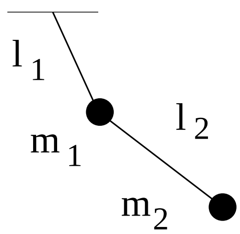
1. Particle moving on a curve, $y = f(x)$, on the (x, y) plane: $\dot{y} = f'(x)\dot{x}$

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - mgf(x) = \frac{m(x)}{2}\dot{x}^2 - mgf(x), \quad m(x) = m[1 + f'^2(x)]$$

2. Pendulum:

$$L = \frac{m}{2}\ell^2\dot{\theta}^2 + mg\ell\cos\theta$$

3. Double pendulum:



$$\begin{aligned} \boldsymbol{x}_1 &= \ell_1 \begin{pmatrix} \sin\theta_1 \\ -\cos\theta_1 \end{pmatrix}, \quad \boldsymbol{x}_2 = \boldsymbol{x}_1 + \ell_2 \begin{pmatrix} \sin\theta_2 \\ -\cos\theta_2 \end{pmatrix} \\ \dot{\boldsymbol{x}}_1 &= \ell_1\dot{\theta}_1 \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix}, \quad \dot{\boldsymbol{x}}_2 = \dot{\boldsymbol{x}}_1 + \ell_2\dot{\theta}_2 \begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} L &= \frac{m_1 + m_2}{2}\ell_1^2\dot{\theta}_1^2 + \frac{m_2}{2}\ell_2^2\dot{\theta}_2^2 + m_2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) \\ &\quad + g(m_1 + m_2)\ell_1\cos\theta_1 + gm_2\ell_2\cos\theta_2 \\ 0 &= m_2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2(-\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2) - g(m_1 + m_2)\ell_1\sin\theta_1 \\ &\quad + (m_1 + m_2)\ell_1^2\ddot{\theta}_1 - m_2\ell_1\ell_2\frac{d}{dt}[\dot{\theta}_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)] \\ 0 &= m_2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2(-\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2) - gm_2\ell_2\sin\theta_2 \\ &\quad - m_2\ell_2^2\ddot{\theta}_2 - m_2\ell_1\ell_2\frac{d}{dt}[\dot{\theta}_1(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)] \end{aligned}$$

C. Noether's theorem

Each continuous symmetry generates a conserved quantity

Symmetry: $\boldsymbol{x}(t) \to \boldsymbol{x}'(t')$, $L(\boldsymbol{x}, \dot{\boldsymbol{x}}) = L(\boldsymbol{x}', \dot{\boldsymbol{x}}') + \dot{\Lambda}(t', \boldsymbol{x}')$

Continuous symmetry: $\exists$ infinitesimal transformations, $\boldsymbol{x} \to \boldsymbol{x} + \epsilon\boldsymbol{f}(t, \boldsymbol{x})$, $t \to t + \epsilon f(t, \boldsymbol{x})$

Conserved quantity: $\dot{F}(\boldsymbol{x}, \dot{\boldsymbol{x}}) = 0$

(a) $\boldsymbol{f} \neq 0$, $f = 0$:

$$L(\boldsymbol{x}, \dot{\boldsymbol{x}}) = L(\boldsymbol{x} + \epsilon\boldsymbol{f}, \dot{\boldsymbol{x}} + \epsilon\partial_t\boldsymbol{f} + \epsilon(\dot{\boldsymbol{x}}\boldsymbol{\partial})\boldsymbol{f}) + \mathcal{O}\left(\epsilon^2\right)$$

A particular variation: $\epsilon \to \epsilon(t)$, $\boldsymbol{x} = \boldsymbol{x}_{cl}$, $\delta S[\epsilon] = \mathcal{O}\left(\epsilon^2\right)$

$$\begin{aligned} \tilde{L}(\epsilon, \dot{\epsilon}) &= L(\boldsymbol{x} + \epsilon\boldsymbol{f}, \dot{\boldsymbol{x}} + \epsilon\partial_t\boldsymbol{f} + \epsilon(\dot{\boldsymbol{x}}\boldsymbol{\partial})\boldsymbol{f} + \dot{\epsilon}\boldsymbol{f}) + \mathcal{O}\left(\epsilon^2\right) \\ &= \epsilon\left[\frac{\partial L}{\partial\boldsymbol{x}}\boldsymbol{f} + \frac{\partial L}{\partial\dot{\boldsymbol{x}}}\partial_t\boldsymbol{f} + \frac{\partial L}{\partial\dot{\boldsymbol{x}}}(\dot{\boldsymbol{x}}\boldsymbol{\partial})\boldsymbol{f}\right] + \frac{\partial L}{\partial\dot{\boldsymbol{x}}}\dot{\epsilon}\boldsymbol{f} + \mathcal{O}\left(\epsilon^2\right) \end{aligned}$$

Euler-Lagrange equation:

$$\underbrace{\frac{\partial\tilde{L}(\epsilon, \dot{\epsilon})}{\partial\epsilon}}_{0} = \frac{d}{dt}\underbrace{\frac{\partial\tilde{L}(\epsilon, \dot{\epsilon})}{\partial\dot{\epsilon}}}_{p_\epsilon}, \quad p_\epsilon = \frac{\partial L}{\partial\dot{\boldsymbol{x}}}\boldsymbol{f}$$

Translations: $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{n}$, $\boldsymbol{n}^2 = 1$. $L = \frac{m}{2}\dot{\boldsymbol{x}}^2 - U(\boldsymbol{T}\boldsymbol{x})$, $T = 1\!\!1 - \boldsymbol{n} \otimes \boldsymbol{n}$, $p_\epsilon = m\dot{\boldsymbol{x}}\boldsymbol{n}$

Rotations: $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{n} \times \boldsymbol{x}$, $\boldsymbol{n}^2 = 1$. $L = \frac{m}{2}\dot{\boldsymbol{x}}^2 - U(|\boldsymbol{x}|)$, $p_\epsilon = m\dot{\boldsymbol{x}}(\boldsymbol{n} \times \boldsymbol{x}) = \boldsymbol{n}(\boldsymbol{x} \times m\dot{\boldsymbol{x}}) = \boldsymbol{n}\boldsymbol{L}$

(b) $\boldsymbol{f} = 0$, $f \neq 0$: $t \to t' = t + \epsilon$, $\boldsymbol{x}(t) \to \boldsymbol{x}(t-\epsilon) = \boldsymbol{x}(t) - \epsilon\dot{\boldsymbol{x}}(t)$, $\delta\boldsymbol{x} = -\epsilon\dot{\boldsymbol{x}}$

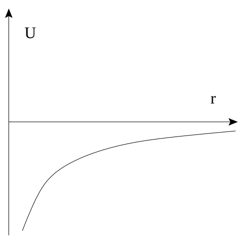
$$
\begin{aligned}
S[\boldsymbol{x}] &= \int_{t_i+\epsilon(t_i)}^{t_f+\epsilon(t_f)} \frac{dt}{1+\dot{\epsilon}} L(\boldsymbol{x}(t-\epsilon), \dot{\boldsymbol{x}}(t-\epsilon)) \\
0 &= -\int_{t_i}^{t_f} dt \left(\epsilon\dot{\boldsymbol{x}}\frac{\partial L}{\partial \boldsymbol{x}} + \frac{d}{dt}\epsilon\dot{\boldsymbol{x}}\frac{\partial L}{\partial \dot{\boldsymbol{x}}} + \dot{\epsilon}L\right) + \epsilon L(\boldsymbol{x}(t), \dot{\boldsymbol{x}}(t))\Big|_{t_i}^{t_f} \\
&= -\int_{t_i}^{t_f} dt \left[\epsilon\dot{\boldsymbol{x}}\left(\frac{\partial L}{\partial \boldsymbol{x}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{x}}}\right) + \dot{\epsilon}L\right] + \epsilon\left(L - \dot{\boldsymbol{x}}\frac{\partial L}{\partial \dot{\boldsymbol{x}}}\right)_{t_i}^{t_f} \\
\epsilon(t) = \epsilon \;\; H &= \frac{\partial L}{\partial \dot{\boldsymbol{x}}}\dot{\boldsymbol{x}} - L \quad \text{is conserved}
\end{aligned}
$$

D. Examples

1. Spherically symmetric potential:

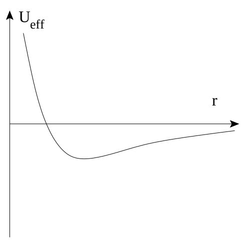
$$
\begin{aligned}
L &= \frac{m}{2}\dot{\boldsymbol{x}}^2 - U(|\boldsymbol{x}|) \\
\boldsymbol{x} &= r\begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}, \quad \dot{\boldsymbol{x}} = \dot{r}\begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} + r\begin{pmatrix} \dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\theta\sin\phi \\ \dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi \\ -\dot{\theta}\sin\theta \end{pmatrix} \\
L &= \frac{m}{2}[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta] - U(r) \\
0 &= \frac{\delta L(x,\dot{x})}{\delta x} - \frac{d}{dt}\frac{\delta L(x,\dot{x})}{\delta \dot{x}} \\
0 &= r^2\dot{\phi}^2\sin\theta\cos\theta - \frac{d}{dt}r^2\dot{\theta} \quad \to \quad \text{planar motion}: \quad \theta = \frac{\pi}{2} \\
0 &= \frac{d}{dt}\frac{\delta L}{\delta\dot{\phi}} = \frac{d}{dt}p_\phi = \frac{d}{dt}mr^2\dot{\phi}\sin^2\theta \quad \to \quad \text{conservation of } p_\phi = mr^2\dot{\phi} = L_z = \ell \\
0 &= -U'(r) - m\ddot{r} + mr(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) = -U'(r) - m\ddot{r} + mr\dot{\phi}^2 \\
m\ddot{r} &= -U'(r) + \frac{\ell^2}{mr^3} = -U'_{eff}(r), \quad U_{eff}(r) = U(r) + \frac{\ell^2}{2mr^2}
\end{aligned}
$$

One-dimensional motion in an effective potential with an centrifugal barrier:



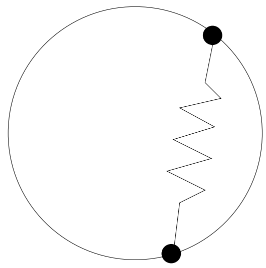
$L = \frac{m}{2}\dot{r}^2 - U_{eff}(r)$

$U(r) = -\frac{e^2}{r}$



- What can one say about the possibility of falling into the center for $U(r) = -\frac{g}{r^n}$ with $n = 2$ or 3?
- Is there circular orbit for $U(r) = -\frac{g_1}{r} - \frac{g_3}{r^3}$?
- What are the stable orbits for $U(r) = -g\frac{e^{-kr}}{r}$?

2. Two particles on a ring: Two point particles of mass m move on a ring of radius r without friction. They are connected by a massless spring of length $\ell < 2r$ and spring constant k.



(a): Find a continuous symmetry and the corresponding conserved quantity.

(b): Solve the equation of motion for small oscillations.

- Lagrangian:

$$L = \frac{m}{2}r^2(\dot{\alpha}^2 + \dot{\beta}^2) - \frac{k}{2}\left(2r\sin\frac{\alpha-\beta}{2} - \ell\right)^2$$

- Symmetry: $\alpha \to \alpha + \phi$, $\beta \to \beta + \phi$ (rotation)

$$L_z = \frac{1}{\phi}\frac{\partial L}{\partial \dot{\boldsymbol{x}}}\delta\boldsymbol{x} = mr^2(\dot{\alpha} + \dot{\beta})$$

- New coordinates: $\Theta = \frac{\alpha+\beta}{2}$, $\chi = \alpha - \beta$, $\Theta \to \Theta + \phi$, $\chi \to \chi$

$$\begin{aligned} L &= \frac{m}{2}r^2(\dot{\alpha}^2 + \dot{\beta}^2) - \frac{k}{2}\left(2r\sin\frac{\alpha-\beta}{2} - \ell\right)^2 \\ &= \frac{m}{2}r^2\left[\left(\dot{\Theta}^2 + \frac{\dot{\chi}}{2}\right)^2 + \left(\dot{\Theta} - \frac{\dot{\chi}}{2}\right)^2\right] - \frac{k}{2}\left(2r\sin\frac{\chi}{2} - \ell\right)^2 \\ &= mr^2\left(\dot{\Theta}^2 + \frac{\dot{\chi}^2}{4}\right) - \frac{k}{2}\left(2r\sin\frac{\chi}{2} - \ell\right)^2 \end{aligned}$$

- Conservation law:

$$p_\Theta = \frac{\partial L}{\partial \dot{\Theta}} = 2mr^2\dot{\Theta} = L_z$$

- E.O.M.

$$\begin{aligned} 0 &= \frac{\delta L(x,\dot{x})}{\delta x} - \frac{d}{dt}\frac{\delta L(x,\dot{x})}{\delta \dot{x}} \\ \delta\Theta:\ 2mr^2\ddot{\Theta} &= 0, \quad \Theta = \Theta_0 + \Omega t \\ \delta\chi:\ \frac{mr^2}{2}\ddot{\chi} &= -kr\left(2r\sin\frac{\chi}{2} - \ell\right)\cos\frac{\chi}{2} \end{aligned}$$

- Small oscillations: $2r\sin\frac{\chi}{2} \approx \ell$, $2r\sin\frac{\chi_0}{2} = \ell$, $\chi = \chi_0 + \epsilon$

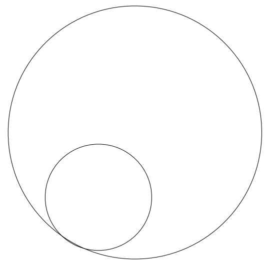
$$\frac{mr^2}{2}\ddot{\epsilon} = -kr\left(2r\sin\frac{\chi_0+\epsilon}{2} - \ell\right)\cos\frac{\chi_0+\epsilon}{2} \approx -kr^2\epsilon\cos^2\frac{\chi_0}{2}$$

$$\omega^2 = 2\frac{k}{m}\cos^2\frac{\chi_0}{2} = 2\frac{k}{m}\left[1-\left(\frac{\ell}{2r}\right)^2\right]$$

Special cases:

$$\begin{aligned} \ell \approx 2r: \quad \omega &\approx 0 \quad \text{Why?} \\ \ell \approx 0: \quad \omega &\approx \sqrt{\frac{2k}{m}} \end{aligned}$$

Sphere in a cylinder: A sphere of radius r and mass m rolls without friction and slipping within a cylinder of radius $R > r$ (in three dimensions!).



(a): Find a continuous symmetry and the corresponding conserved quantity.

(b): Solve the equation of motion for small oscillations.

- Cordinate: (Θ, z)
- Lagrangian: $\mu = \frac{m}{4\pi r^2} = \frac{dm}{d\sigma}$, cylindrical coordinates (r, ϕ, z),

$$\begin{aligned} K &= \frac{\mu}{2}\underbrace{r\int_{-r}^{r} dz\sqrt{1+(\partial_z\sqrt{r^2-z^2})^2}\int_0^{2\pi}}_{\int d\sigma} d\alpha\{r^2\dot\Theta^2[(1+\sin\alpha)^2+\cos^2\alpha]+\dot z^2\} \\ &= \frac{m}{2}\left[2r^2\dot\Theta^2\int_0^{2\pi}\frac{d\alpha}{2\pi}(1+\sin\alpha)+\dot z^2\right] \\ &= \frac{m}{2}(2r^2\dot\Theta^2+\dot z^2) \\ U &= mgR(1-\cos\phi), \quad R\phi = r\Theta \\ L &= \frac{m}{2}\dot z^2 + mr^2\dot\Theta^2 - mgR\left(1-\cos\Theta\frac{r}{R}\right) \end{aligned}$$

- Symmetry: $z \to z + a$, $p_z = \frac{\delta L}{\delta \dot z} = m\dot z$
- E.O.M.

$$\begin{aligned} 0 &= \frac{\delta L(x,\dot x)}{\delta x} - \frac{d}{dt}\frac{\delta L(x,\dot x)}{\delta \dot x} \\ \delta\Theta: \ 2mr^2\ddot\Theta &= -mgr\sin\Theta\frac{r}{R} \\ \delta z: \ m\ddot z &= 0, \quad z = z_0 + v_z t \end{aligned}$$

- Small oscillations: $\Theta \approx 0$

$$\ddot\Theta = -\frac{mgr}{2mr^2}\frac{r}{R}\Theta = -\frac{g}{2mR}\Theta, \quad \omega = \sqrt{\frac{g}{2mR}}$$

E. Relativistic particle

- Lorentz invariant action:

$$\begin{aligned} S &= -mc\int_{s_i}^{s_f} ds \\ &= \int_{s_i}^{s_f} d\tau \underbrace{(-mc)\sqrt{\frac{dx^\mu}{d\tau}g_{\mu\nu}\frac{dx^\mu}{d\tau}}}_{L_\tau} \end{aligned}$$

- E.O.M.: $x^\mu(\tau) \to x^\mu(\tau) + \delta x^\mu(\tau)$ $(\tau \neq s)$,

$$\begin{aligned} 0 &= -mc\frac{d}{d\tau}\frac{\frac{dx^\mu}{d\tau}}{\sqrt{\frac{dx^\rho}{d\tau}g_{\rho\nu}\frac{dx^\mu}{d\tau}}} = -mc\frac{\frac{d^2x^\mu}{d\tau^2}}{\sqrt{\frac{dx^\rho}{d\tau}g_{\rho\nu}\frac{dx^\mu}{d\tau}}} + mc\frac{\frac{dx^\mu}{d\tau}\frac{d^2x^\rho}{d\tau^2}g_{\rho\nu}\frac{dx^\mu}{d\tau}}{\sqrt{\left(\frac{dx^\rho}{d\tau}g_{\rho\nu}\frac{dx^\mu}{d\tau}\right)^3}} \\ &= -mc\ddot{x}^\mu \quad (\tau = s) \end{aligned}$$

- The four-momentum:

$$p^\mu = -\frac{\partial L}{\partial \dot{x}_\mu} = mc\dot{x}^\mu$$

- Nonrelativistic notation: $x^\mu = (ct, \boldsymbol{x})$,

$$\begin{aligned} S &= \int_{s_i}^{s_f} d\tau(-mc)\sqrt{\frac{dx^\mu}{d\tau}g_{\mu\nu}\frac{dx^\mu}{d\tau}} \\ &= \int_{t_i}^{t_f} dt \underbrace{(-mc^2)\sqrt{1-\frac{\boldsymbol{v}^2}{c^2}}}_{L_t} = -mc^2(t_f - t_i) + \int_{x_i}^{x_f} dt\left[\frac{m}{2}\boldsymbol{v}^2 + \mathcal{O}\left(\frac{v^4}{c^2}\right)\right] \\ \boldsymbol{p} &= \frac{\partial L}{\partial \boldsymbol{v}} = \frac{m\boldsymbol{v}}{\sqrt{1-\frac{v^2}{c^2}}} \\ E &= \frac{\partial L}{\partial \dot{\boldsymbol{x}}}\dot{\boldsymbol{x}} - L = \boldsymbol{p}\boldsymbol{v} - L \\ &= \frac{m\left[v^2 + c^2\left(1-\frac{v^2}{c^2}\right)\right]}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} = mc^2 + \frac{v^2}{2m} + \mathcal{O}\left(\frac{v^4}{c^2}\right) \end{aligned}$$

Motion in a spherically symmetric potential:

- Coordinates: $x^\mu = (ct, r, \theta, \phi)$

- Action:

$$\begin{aligned} S &= -mc\int ds\sqrt{\dot{x}^\mu g_{\mu\nu}\dot{x}^\nu} - \int dtU(r) = -\int ds[mc\sqrt{\dot{x}^\mu g_{\mu\nu}\dot{x}^\nu} + \dot{t}U(r)] \\ L &= -mc\sqrt{\dot{x}^\mu g_{\mu\nu}\dot{x}^\nu} - \dot{t}U(r) \\ &= -mc\sqrt{c^2\dot{t}^2 - \dot{r}^2 - r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)} - \dot{t}U(r) \end{aligned}$$

- $\delta\theta$:

$$r^2 \sin\theta\cos\theta\dot\phi^2 = \frac{d}{ds} r^2\dot\theta \quad \to \quad \theta(s) = \frac{\pi}{2}$$

- Cyclic coordinates: t, ϕ:

$$\begin{aligned} \frac{\partial L}{\partial \dot t} &= -mc^3\dot t - U(r) = -mc^2 - E \quad \to \quad \dot t = \frac{(mc^2+E-U)^2}{m^2c^6}, \\ \frac{\partial L}{\partial \dot\phi} &= mcr^2\dot\phi = \ell, \end{aligned}$$

- Energy equation: $1 = \dot x^2$

$$1 = c^2\dot t^2 - \dot r^2 - r^2\dot\phi^2 = \frac{(mc^2+E-U)^2}{m^2c^4} - \dot r^2 - \frac{\ell^2}{m^2c^2r^2}$$

- Radial equation of motion:

$$\begin{aligned} \dot r^2 &= \dot t^2\left(\frac{dr}{dt}\right)^2 = \frac{(mc^2+E-U)^2}{m^2c^4} - \frac{\ell^2}{m^2c^2r^2} - 1 \\ \left(\frac{dr}{dt}\right)^2 &= \frac{m^2c^6}{(mc^2+E-U)^2}\left[\frac{(mc^2+E-U)^2}{m^2c^4} - \frac{\ell^2}{m^2c^2r^2} - 1\right] \\ &= c^2\left[1 - \frac{\frac{\ell^2c^2}{r^2} + m^2c^4}{(mc^2+E-U)^2}\right] \\ &= c^2\left[1 - \frac{\frac{\ell^2}{m^2c^2r^2} + 1}{(1+\frac{E-U}{mc^2})^2}\right] \end{aligned}$$

- Nonrelativistic limit, $\frac{v}{c} \to 0$, in the leading $\mathcal{O}\left((\frac{v}{c})^0\right)$ order: $(U, E = \mathcal{O}\left((\frac{v}{c})^0\right))$:

$$\begin{aligned} \left(\frac{dr}{dt}\right)^2 &\approx c^2\left[1 - \left(\frac{\ell^2}{m^2c^2r^2} + 1\right)\left(1 - 2\frac{E-U}{mc^2}\right)\right] \approx c^2\left(-\frac{\ell^2}{m^2c^2r^2} + 2\frac{E-U}{mc^2}\right) \\ &= -\frac{\ell^2}{m^2r^2} + \frac{2}{m}(E-U) \\ E &\approx \frac{m}{2}\left(\frac{dr}{dt}\right)^2 + \frac{\ell^2}{2mr^2} + U \end{aligned}$$

- Further problems:

  1. $\mathcal{O}\left((\frac{v}{c})^2\right)$?

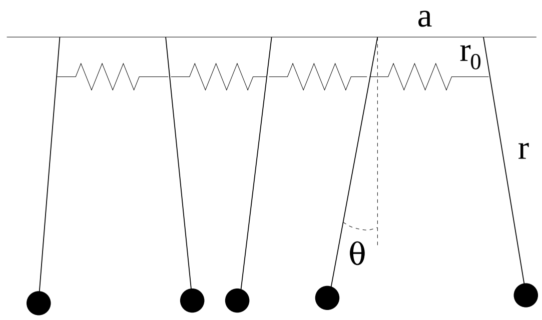
  2. Find the equation of motion of the action

$$S = -mc\int ds\sqrt{\dot x^\mu g_{\mu\nu}\dot x^\nu} - \frac{1}{c}\int ds U(r)$$

VII. FIELD THEORIES

Mechanical analogy: $\theta_n(t) \to \Phi\theta_n(t) = \phi(t, x_n)$, $\Phi = r_0\sqrt{ak}$, $c = a\frac{r_0}{r}\sqrt{\frac{k}{m}}$, $\lambda = \frac{gr}{a}$

$$\begin{aligned}
L &= \sum_n \left[\frac{mr^2}{2}\dot\theta_n^2 - \frac{kr_0^2}{2}(\theta_{n+1}-\theta_n)^2 + gr\cos\theta_n\right]\\
&= a\sum_n \left[\frac{1}{2c^2}(\partial_t\phi_n)^2 - \frac{1}{2}\left(\frac{\phi_{n+1}-\phi_n}{a}\right)^2 + \lambda\cos\frac{\phi_n}{\Phi}\right]\\
&\to \int dx \left[\frac{1}{2c^2}(\partial_t\phi(x))^2 - \frac{1}{2}(\partial_x\phi(x))^2 + \lambda\cos\frac{\phi(x)}{\Phi}\right]\\
S &= \int dtdx \left[\frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) + \lambda\cos\frac{\phi(x)}{\Phi}\right]
\end{aligned}$$



A. Equation of motion

Auxiliary conditions: $\phi(t_i,\boldsymbol{x}) = \phi_i(\boldsymbol{x})$, $\phi(t_f,\boldsymbol{x}) = \phi_f(\boldsymbol{x})$

Action:

$$S[\phi(\cdot)] = \int_V \underbrace{dtd^3x}_{\frac{1}{c}d^4x} L(\phi,\partial\phi)$$

Variation:

$$\phi(x) \to \phi(x) + \delta\phi(x), \qquad \delta\phi(t_i,\boldsymbol{x}) = \delta\phi(t_f,\boldsymbol{x}) = 0$$

$$\begin{aligned}
\delta S &= \int_V dx \left(\frac{\partial L(\phi,\partial\phi)}{\partial\phi_a}\delta\phi_a + \frac{\partial L(\phi,\partial\phi)}{\partial\partial_\mu\phi_a}\delta\partial_\mu\phi_a\right) + \mathcal{O}\left(\delta^2\phi\right)\\
&= \int_V dx \left(\frac{\partial L(\phi,\partial\phi)}{\partial\phi_a}\delta\phi_a + \frac{\partial L(\phi,\partial\phi)}{\partial\partial_\mu\phi_a}\partial_\mu\delta\phi_a\right) + \mathcal{O}\left(\delta^2\phi\right)\\
&= \int_{\partial V} ds^\mu\delta\phi_a\frac{\partial L(\phi,\partial\phi)}{\partial\partial_\mu\phi_a} + \int_V dx\delta\phi_a\left(\frac{\partial L(\phi,\partial\phi)}{\partial\phi_a} - \partial_\mu\frac{\partial L(\phi,\partial\phi)}{\partial\partial_\mu\phi_a}\right) + \mathcal{O}\left(\delta^2\phi\right)
\end{aligned}$$

Surface contributions:

$$\begin{aligned}
\mu = 0 \ &: \int_{\partial V} ds^0\delta\phi_a\frac{\partial L(\phi,\partial\phi)}{\partial\partial_0\phi_a} = \int_{t=t_f} d^3x \underbrace{\delta\phi_a}_{0}\frac{\partial L(\phi,\partial\phi)}{\partial\partial_0\phi_a} - \int_{t=t_i} d^3x \underbrace{\delta\phi_a}_{0}\frac{\partial L(\phi,\partial\phi)}{\partial\partial_0\phi_a} = 0\\
\mu = j \ &: \int_{\partial V} ds^j\delta\phi_a\frac{\partial L(\phi,\partial\phi)}{\partial\partial_j\phi_a} = \int_{x_j=\infty} ds^j\delta\phi_a\frac{\partial L(\phi,\partial\phi)}{\partial\partial_j\phi_a} - \int_{x_j=-\infty} ds^j\delta\phi_a\frac{\partial L(\phi,\partial\phi)}{\partial\partial_j\phi_a} \quad \text{ignored}
\end{aligned}$$

Euler-Lagrange equation:

$$\frac{\partial L(\phi,\partial\phi)}{\partial\phi_a} - \partial_\mu\frac{\partial L(\phi,\partial\phi)}{\partial\partial_\mu\phi_a} = 0.$$

B. Wave equations for a scalar particle

Schrödinger equation:

$$i\hbar\partial_t\psi(t,\boldsymbol{x}) = \left[-\frac{\hbar^2}{2m}\Delta + U(\boldsymbol{x})\right]\psi(t,\boldsymbol{x}), \quad L = \psi^*\left[i\hbar\partial_t + \frac{\hbar^2}{2m}\Delta - U(\boldsymbol{x})\right]\psi$$

Relativistic generalisation: $\hat{p}_\mu = (\frac{E}{c}, -\boldsymbol{p}) = -\frac{\hbar}{i}\partial_\mu$, $\hat{p}^\mu = (\frac{E}{c}, \boldsymbol{p}) = i\hbar\partial^\mu$,

Klein-Gordon equation: $p^\mu = mc\dot{x}^\mu$, $p^2 = m^2c^2$:

$$\begin{aligned} 0 &= (\hat{p}^2 - m^2c^2)\phi = -\hbar^2\left(\partial_\mu\partial^\mu + \frac{m^2c^2}{\hbar^2}\right)\phi, \\ 0 &= \left(\Box + \frac{1}{\lambda_C^2}\right)\phi, \quad \lambda_C = \frac{\hbar}{mc} \quad \text{(Compton wavelength)} \end{aligned}$$

with interaction:

$$\begin{aligned} \text{real} \quad L &= \frac{1}{2}(\partial\phi)^2 - \frac{1}{2\lambda_C^2}\phi^2 - U(\phi) \\ \text{complex} \quad L &= \partial\phi^*\partial\phi - \frac{1}{\lambda_C^2}\phi^*\phi - V(\phi^*\phi) \end{aligned}$$

Free particle: $U(\phi) = \frac{m^2c^2}{2\hbar^2}\phi^2$, $V(\phi^*\phi) = \frac{m^2c^2}{\hbar^2}\phi^*\phi$, $\phi(x) = e^{-\frac{i}{\hbar}p_\mu x^\mu}$,

$$\begin{aligned} 0 &= \left(\partial_\mu\partial^\mu + \frac{m^2c^2}{\hbar^2}\right)e^{-\frac{i}{\hbar}p_\mu x^\mu} \\ 0 &= p^2 - \frac{m^2c^2}{\hbar^2} \\ \phi(x) &= e^{\mp i p_\mu x^\mu}, \quad p_0 = \omega_p = \sqrt{\frac{m^2c^2}{\hbar^2} + \boldsymbol{p}^2} \end{aligned}$$

C. Electrodynamics

Charge: $x_a^\mu(s)$, $a = 1, \ldots, n$, EM field: $A^\mu(x)$

Charge:

$$\begin{aligned} S_{ch} &= -\int_{x_i}^{x_f}\left(mcds + \frac{e}{c}A_\mu dx^\mu\right) = \int_{\tau_i}^{\tau_f} L_\tau d\tau \\ L_\tau &= -mc\sqrt{x'^2} - \frac{e}{c}A_\mu(x)x'^\mu \end{aligned}$$

Gauge transformation: $A_\mu(x) \to A_\mu(x) + \partial_\mu\Phi(x)$

Gauge invariance: $\Delta S = \frac{e}{c}[\Phi(x_i) - \Phi(x_f)]$ drops out from the equations of motion

Euler-Lagrange equation:

$$\begin{aligned} 0 &= \frac{\partial L_\tau}{\partial x^\mu} - \frac{d}{d\tau}\frac{\partial L_\tau}{\partial x'^\mu} = -\frac{e}{c}\partial_\mu A_\nu(x)x'^\nu + mc\frac{d}{d\tau}\frac{x'_\mu}{\sqrt{x'^2}} + \frac{e}{c}\frac{d}{d\tau}A_\mu(x) \\ \tau \to s: \quad mc\ddot{x}_\nu &= \frac{e}{c}F_{\mu\nu}\dot{x}^\nu \end{aligned}$$

Field strength tensor:

$$F_{\mu\nu} = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x).$$

Electric current: $\frac{dx^0(s)}{ds} > 0$,

$$j^\mu(x) = c\sum_a \int ds\delta^{(4)}(x - x_a(s))\dot{x}^\mu$$

$$\begin{aligned}
&= c\sum_a \int ds\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}_a(s))\delta(x^0-x_a^0(s))\dot{x}^\mu, \quad ds=\frac{d\dot{x}^0(s)}{|\dot{x}^0|} \\
&= c\sum_a \delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}_a(s))\frac{\dot{x}^\mu}{|\dot{x}^0|}, \quad \frac{\dot{x}^\mu}{|\dot{x}^0|}=\frac{\frac{dx^\mu}{ds}}{\frac{dct}{ds}}=\frac{1}{c}\frac{dx^\mu}{dt} \\
&= \underbrace{\sum_a \delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}_a(s))}_{\rho(\boldsymbol{x})}\frac{dx^\mu}{dt}=(c\rho,\rho\boldsymbol{v})
\end{aligned}$$

Current conservation:

$$\partial_\mu j^\mu = c\partial_0\rho+\nabla\cdot\boldsymbol{j}=\sum_a e_a[-\boldsymbol{v}_a(t)\boldsymbol{\nabla}\delta(\boldsymbol{x}-\boldsymbol{x}_a(t))+\boldsymbol{\nabla}\delta(\boldsymbol{x}-\boldsymbol{x}_a(t))\boldsymbol{v}_a(t)]=0$$

EM field:

Lagrangian:

1. $L_A=\mathcal{O}\left((\partial_0 A_\mu)^2\right)$
2. invariance de Lorentz
3. invariance de jauge

$$L_A=-\frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu}$$

Full action:

$$\begin{aligned}
S_{EM} &= -\sum_a\left[mc\int ds\sqrt{\dot{x}_a^\mu g_{\mu\nu}\dot{x}_a^\nu}+\frac{e}{c}\int A_\mu(x)dx^\mu\right]-\frac{1}{16\pi c}\int F^{\mu\nu}F_{\mu\nu}dx \\
&= -mc\sum_a\int ds\sqrt{\dot{x}_a^\mu g_{\mu\nu}\dot{x}_a^\nu}-\frac{1}{c}\int\left[\frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu}+\frac{e}{c}j^\mu A_\mu\right]dx
\end{aligned}$$

$$\begin{aligned}
0 &= \frac{\delta L}{\delta A_\mu}-\partial_\nu\frac{\delta L}{\delta\partial_\nu A_\mu}, \qquad F^{\mu\nu}F_{\mu\nu}=2\partial_\mu A_\nu\partial^\mu A^\nu-2\partial_\mu A_\nu\partial^\nu A^\mu \\
&= -\frac{e}{c}j^\mu+\frac{1}{4\pi}\partial_\nu(\partial^\nu A^\mu-\partial^\mu A^\nu) \\
\frac{1}{4\pi}\partial_\nu F^{\nu\mu} &= \frac{e}{c}j^\mu
\end{aligned}$$

Bianchi identity : $\partial_\mu\partial_\nu A_\rho=\partial_\nu\partial_\mu A_\rho$

$$\partial_\rho F_{\mu\nu}+\partial_\nu F_{\rho\mu}+\partial_\mu F_{\nu\rho}=0$$

Maxwell: $(\boldsymbol{E},\boldsymbol{H})\to A_\mu$, the first unification in physics, $A^\mu=(\phi,\boldsymbol{A})$, $A_\mu=(\phi,-\boldsymbol{A})$

$$\begin{aligned}
\boldsymbol{E} &= -\partial_0\boldsymbol{A}-\nabla\phi=-\frac{1}{c}\partial_t\boldsymbol{A}-\nabla\phi, \\
\boldsymbol{H} &= \nabla\times\boldsymbol{A}.
\end{aligned}$$

Inversion:

$$\epsilon_{jk\ell}H_\ell = \epsilon_{jk\ell}\epsilon_{\ell mn}\nabla_m A_n = (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km})\nabla_m A_n = \nabla_j A_k - \nabla_k A_j = -F_{jk}$$

Field strength:

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -H_z & H_y \\ -E_y & H_z & 0 & -H_x \\ -E_z & -H_y & H_x & 0 \end{pmatrix}, \quad F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -H_z & H_y \\ E_y & H_z & 0 & -H_x \\ E_z & -H_y & H_x & 0 \end{pmatrix}$$

Dual field strength:

$$\begin{aligned}
\tilde{F}_{\mu\nu} &= \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} \\
\tilde{F}_{0j} &= -\frac{1}{2}\epsilon_{jk\ell}F^{k\ell} = \frac{1}{2}\epsilon_{jk\ell}\epsilon_{k\ell m}H_m = H_j \quad (\epsilon_{123} = -\epsilon^{123} = -1) \\
\tilde{F}_{jk} &= -\epsilon_{jk\ell}F^{\ell 0} = \epsilon_{jk\ell}E_\ell \\
\tilde{F}_{\mu\nu} &= \begin{pmatrix} 0 & H_x & H_y & H_z \\ -H_x & 0 & E_z & -E_y \\ -H_y & -E_z & 0 & E_x \\ -H_z & E_y & -E_x & 0 \end{pmatrix}, \quad \tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -H_x & -H_y & -H_z \\ H_x & 0 & E_z & -E_y \\ H_y & -E_z & 0 & E_x \\ H_z & E_y & -E_x & 0 \end{pmatrix}
\end{aligned}$$

Inhomogeneous Maxwell equations:

$$\begin{aligned}
\frac{4\pi}{c}j^0 = \nabla_j F^{j0}, &\quad \rightarrow \quad 4\pi\rho = \boldsymbol{\nabla E} \\
\frac{4\pi}{c}j^k = \partial_0 F^{0k} + \nabla_j F^{jk} &\quad \rightarrow \quad \frac{4\pi}{c}\boldsymbol{j} = -\frac{1}{c}\partial_t \boldsymbol{E} + \boldsymbol{\nabla} \times \boldsymbol{H}
\end{aligned}$$

Homogeneous Maxwell equations:

$$\begin{aligned}
0 &= \partial_\mu \tilde{F}^{\mu\nu} \\
0 &= \nabla_j \tilde{F}^{j0} = \boldsymbol{\nabla H}, \\
0 &= \partial_0 \tilde{F}^{0k} + \nabla_j \tilde{F}^{jk} = -\left(\frac{1}{c}\partial_t \boldsymbol{H} + \boldsymbol{\nabla} \times \boldsymbol{E}\right)
\end{aligned}$$