# Special relativity

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## I. PHYSICAL LAWS DEPEND ON THE SCALE OF OBSERVATION

One of the important developments of the last decade in physics comes from the application of the renormalization group method, namely that the result of measurements depend on the dimensional parameters, briefly the scales, of the observation.

**Charge in a polarizable medium:** Let us place a particle of charge $q > 0$ into a classical, polarizable medium and measure its charge from distance $r$ by observing the Coulomb force, acting at a test charge $q' \ll q$. The simple Coulomb law,

$$F_C(r) = \frac{qq'}{r^2}, \tag{1}$$

is not valid owing to the polarization cloud around the charge $q$, a a spherically symmetric distribution of dipoles where the dipoles turn their negative charge excess end towards the charge $q$ as seen in Fig. 1. The charge of the particle is not well defined due to this polarization cloud which follows the charge, as long as it moves slowly. It is known that the Coulomb force of a spherically symmetric charge distribution at distance $r$ can be calculated by concentrating all charge within a sphere or radius $r$ to the center. The dipoles falling completely within the sphere give vanishing contributions in this process and only the dipoles, cut by the surface of the sphere should be counted. They tend to give negative contributions hence the original charge is screened. This can be taken into account by defining an effective "running charge" $q(r)$ by parameterizing the measured electrostatic force as

$$F_{measured}(r) = \frac{q(r)q'}{r^2}. \tag{2}$$

The running of the charge is not limited to classical polarizable medium. A charge, inserted into the vacuum of quantum electrodynamics, generates virtual electron-positron pairs within the Compton wavelength of the charge, realizing a similar phenomenon.

**Mass of a ball:** Let us consider a ball of mass $M$ moving with velocity $v$ within a viscous fluid. The mass of the ball is now ill defined due to the quantity of fluid, dragged along. A possible strategy to measure its mass is to find out the energy of the fluid as the function of velocity $v^2$ (the energy is symmetric with respect to spatial inversion, $v \to -v$) and to write

$$E(v) = E(v_0) + (v - v_0)\frac{dE(v_0)}{dv} + \frac{(v-v_0)^2}{2}\frac{d^2E(v_0)}{dv^2} + \cdots \tag{3}$$

leading to

$$M(v_0) = \frac{d^2E(v_0)}{dv^2} \tag{4}$$

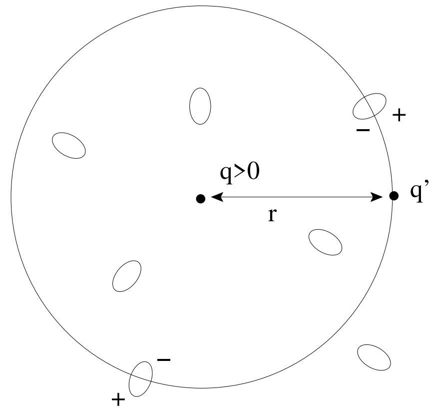



FIG. 1: The spherically symmetrical polarization cloud of a charge $q$.

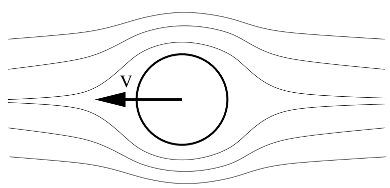



FIG. 2: The idealized flow pattern around a moving ball.

The general lesson is that the physical constants are replaced by effective, "running" values, $q(r)$, $M(v)$, etc. due to the interaction of the observed system with its environment and their particular approximative value correspond only to a plateau in a certain scale window as shown in Fig. 3 for the charge. Furthermore the physical laws, relating the physical "constants", are scale dependent, too.

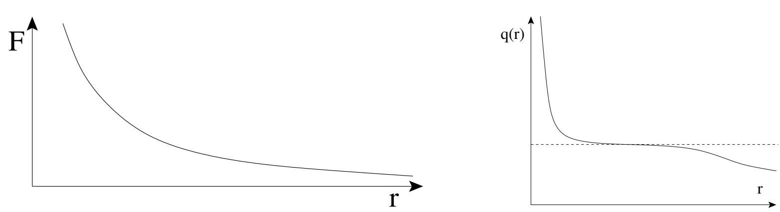



FIG. 3: The qualitative distance dependence of the effective force, acting between two static point charges and the running charge, $q(r)$, extracted from quantum electrodynamics.

The scale dependence of the physical laws can nicely be demonstrated by rearranging the physical phenomena according to their length scale, as shown in Fig. 4. Our problem is that our intuition, developed in childhood, belongs approximatively to the scale window $10^{-3}m < \ell < 10^{3}m$ hence we use the symbolic language of mathematics to cover the range of phenomenas we encounter by using the microscope and the telescope.

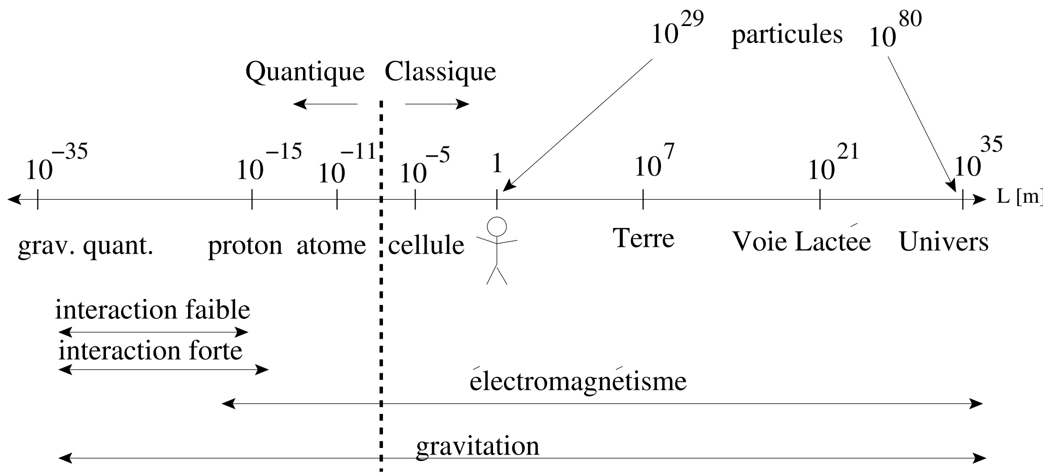



FIG. 4: Few characteristic phenomenas seen by changing the spatial resolution of the observation.

The structure of the physical laws is radically different on the two sides of the dashed line of Fig. 4, separating microscopic and macroscopic physics. Few surprising elements of the microscopic side are the following:

- The interpretation of the Heisenberg uncertainty relation is that the reality can not fully be known due to the disturbing feature of the observations.
- Quantum mechanics can be interpreted as the optimized and consistent treatment of partial information.
- A quantum state consists of a list of virtual realities, similar to a phone book where the name identify the virtual reality and the telephone number gives the corresponding probability amplitude.

An intriguing feature of the crossover regime between the microscopic and macroscopic world is its closeness to the natural length scale of life. The elementary unit of life is the protein, an organic macromolecule type. There are $4^{165}$ possible RNA chain of 165 nucleotics however only 4500 of them are found in living organisms. The life was believed to originate from a primordial soup, containing smaller organic fraction of the proteins. However a soup, containing one sample of each of the possible ribosome would have the mass $M = 2 \times 10^{77} kg \sim M_{Universe} \times 10^{25}$. Hence there must be a very efficient selection mechanism to provide the necessary elements to start the life. 450 randomly chosen proteins transport charges at the quantum-classical crossover, formerly believed to be at shorter scale. The choice of a particularly efficient transport mechanism, lying at the border of the macroscopic and the microscopic world, suggests an important role of quantum effects in life, confirmed by the identification of several other of key mechanisms of living organisms within the quantum domain.

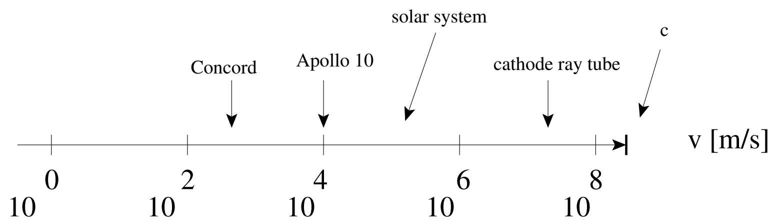



FIG. 5: Few object as the function of their characteristic velocity.

Let us now make a step towards special relativity by placing few objects onto Fig. 5 according to their characteristic velocities. Special relativity is about the velocity-dependence of the physical laws. There seems to be no higher velocity in nature than the speed of light. There is a remarkable exception, arising from the entanglement of quantum states but this can not be used to send preselected informations thereby preventing us to exploit it. One finds two regimes in the mechanics of a particle of mass $m$, separated by the characteristic momentum scale, $mc$, on the momentum axes, shown in Fig. 6. Our presentation of special relativity starts with demonstrating that Newton's equation, $F = ma$, is modified around the momentum range $p \sim mc$, at $v \sim c = 2.9979 \cdot 10^8 m/s$, $c$ being the speed of light. To appreciate the distance of this regime of our daily life imagine an object in a free fall, $v = 9.8t[MKS]$, its velocity after a year is $v = 9.8 \times 365 \times 24 \times 3600 \approx 3 \times 10^8 m/s$

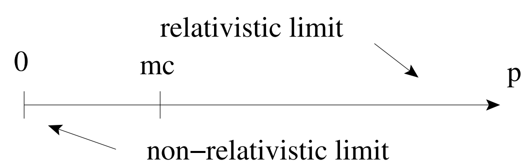



FIG. 6: The relativistic and the non-relativistic regimes on the momentum scale.

## II. A CONFLICT AND ITS SOLUTION

The special relativity grew out from the resolution of a conflict, created by the Michelson-Morley experiment.

### A. Reference frame and Galilean symmetry

The simplicity of Newton's second law can be summarized by Galilean the relativity principle. One introduces the reference frames as the family of the space and time coordinate systems in which the laws of mechanics assumes the same form. If one considers free particles only, $m\ddot{\boldsymbol{x}}(t) = 0$, then the reference frames are related to each other by the transformations

$$(t, \boldsymbol{x}) \to (t', \boldsymbol{x}') = (t, R\boldsymbol{x} + t\boldsymbol{v} + \boldsymbol{x}_0), \tag{5}$$

$R$ being a rotational matrix, $R^{\text{tr}}R = \mathbb{1}$. The transformation with $R = 1$ and $\boldsymbol{x}_0 = 0$, $\boldsymbol{x} \to \boldsymbol{x} + t\boldsymbol{v}$ is called Galilean boost. The Galilean relativity principle is the generalization of this result for interactive particle and states that the mechanical laws are identical in each reference frame. As an example on may consider an object, falling freely from the the mast of a ship, moving with a constant velocity, ends up at the bottom of the mast on a ship.

The time remains unchanged and is considered absolute in Newton's mechanics. There is an important corollary of the absolute nature of the time, namely the velocities add up. In fact, the velocity transforms as

$$\dot{\boldsymbol{x}} \to \dot{\boldsymbol{x}}' = \frac{d}{dt}(\boldsymbol{x} - t\boldsymbol{v}) = \dot{\boldsymbol{x}} - \boldsymbol{v} \tag{6}$$

under Galilean boost.

### B. Limiting velocity

The breakdown of Newton's theory at velocities, comparable with the speed of light, is nicely demonstrated in the experiment, W. Bertozzi, Am. J. Phys. **32** 551 (1964), where a Van de Graaff and a linear accelerator are used in tandem to accelerate electron pulses to measure their time of flight, c.f. Fig. 7.

The result, shown in Table I, indicates that the velocity of the pulses seem to be saturated around the speed of light which appears a a limiting velocity. One encounters a strong violation of Newton's theory in this regime which predicts that the increase of the kinetic energy, $E \to 30E$, correspond to the same rate of change of the velocity square. The final velocity square, plotted on Fig. 8

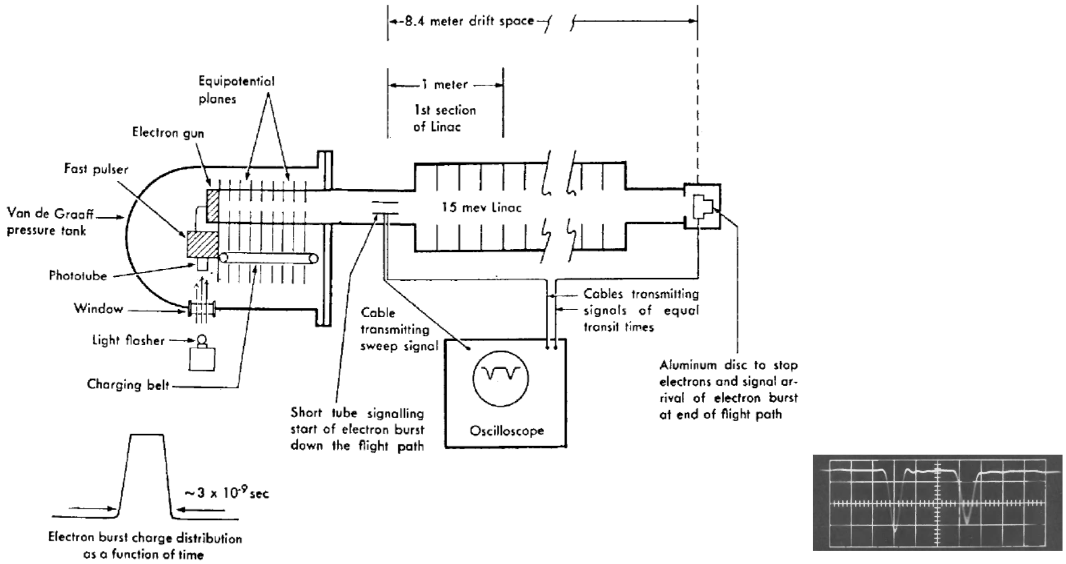



FIG. 7: The measurement of the time of flight of an electron pule within a linear accelerator. The vertical motion of the oscilloscope curve is triggered by the entering and the leaving of the pulses, the difference being $10^{-8}s$ between the vertical lines.

| Energy $K[MeV]$ | Time of flight$t[10^{-8}s]$ | Velocity $v[10^8 m/s]$ | Velocity square $v^2[10^{16}m^2/s^2]$ |
|---|---|---|---|
| 0.5 | 3.23 | 2.60 | 6.8 |
| 1.0 | 3.08 | 2.73 | 7.5 |
| 1.5 | 2.92 | 2.88 | 8.3 |
| 4.5 | 2.84 | 2.96 | 8.8 |
| 15 | 2.80 | 3.00 | 9.0 |

TABLE I: The result of the time of flight measurement.

### C. Particle or wave?

The physics has developed enormously since Galileo, in particular a strong interest of the XIX.-th century physicists was oriented towards the nature of light:

1. Thomas Young (1801-04): measurement of interference

2. Augustin-Jean Fresnel (1818): explication of interference, diffraction, polarization

3. James Clerk Maxwell (1861): electrodynamics, its relation to light realized later

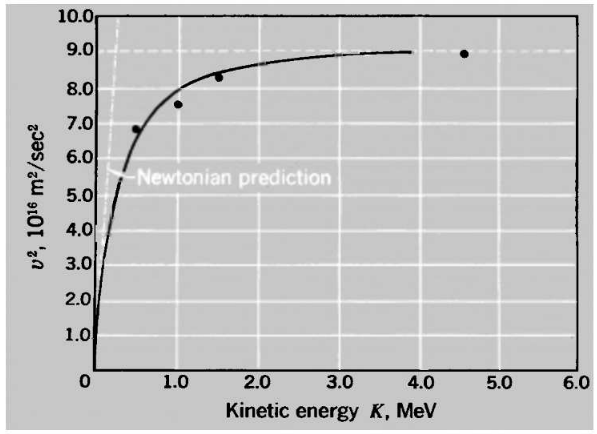



FIG. 8: The violation of the Newtonian kinetic energy expression.

On the one hand, the wave nature of light was supported by the phenomenas of diffraction, interference and polarization. On the other hand, the energy-momentum content of the light suggested a particle nature. The mechanical models, developed until 1850, seemed acceptable however the speed of light was unreasonably high for a particle.

The issue was then approached from a different point of view, by considering the emission of light by a moving source. According to the particle nature one expects $\boldsymbol{v}_{part} = \boldsymbol{v}_{source} + \boldsymbol{v}_{light}$. The wave-like nature suggests $\boldsymbol{v}_{wave}$ which is independent of the speed of the source and is given only by the medium in which the light propagates, called ether. This alternative seemed more realistic however leads immediately to the difficult question whether we can measure our velocity with respect to the ether? A unexpectedly precise and clear answer was given was extracted form the interpretation of the experiment, carried out by Michelson and Morley.

### D. Propagation of the light

The measurement of Michelson (1881) is a surprising example of an observation which despite its simplicity changed our way of looking at physics. He studied the interference between two parts of a monochromatic light beam, shown on Fig. 9. Let us suppose that the laboratory where the experiment is carried out moves with velocity $v$ with respect to the ether in the direction of one of the arms of the interferometer. The time of flight along the arm, parallel to the motion with respect to the ether is

$$t_\parallel = \frac{\ell_\parallel}{c-v} + \frac{\ell_\parallel}{c+v} = \frac{2c\ell_\parallel}{c^2-v^2} = \frac{\frac{2\ell_\parallel}{c}}{1-\frac{v^2}{c^2}}. \tag{7}$$

The corresponding time of the perpendicular arm satisfies the equation

$$\left(\frac{ct_\perp}{2}\right)^2 = \ell_\perp^2 + \left(\frac{vt_\perp}{2}\right)^2, \tag{8}$$

and is given by

$$t_\perp = \frac{2\ell_\perp}{\sqrt{c^2-v^2}}. \tag{9}$$

The difference of the time of flights turns out to be

$$\Delta t(\ell_\parallel, \ell_\perp) = t_\parallel - t_\perp = \frac{2}{c}\frac{\ell_\parallel - \ell_\perp\sqrt{1-\frac{v^2}{c^2}}}{1-\frac{v^2}{c^2}}. \tag{10}$$

A rotation by $90^0$ within few minutes exchanges the two arms and introduces a minus sign,

$$\Delta t'(\ell_\parallel, \ell_\perp) = -\Delta t(\ell_\perp, \ell_\parallel) = \frac{2}{c}\frac{\ell_\parallel\sqrt{1-\frac{v^2}{c^2}} - \ell_\perp}{1-\frac{v^2}{c^2}}, \tag{11}$$

and the change of the interference phase during the rotation is therefore

$$\tau = \Delta t(\ell_\parallel, \ell_\perp) - \Delta t'(\ell_\parallel, \ell_\perp) \approx (\ell_\parallel + \ell_\perp)\frac{v^2}{c^3}. \tag{12}$$

The mirrors are not exactly perpendicular hence interference rings appear in the rejoined beam. They monitored carefully the possible shift of the rings during the rotation. The number of shifted rings, $\Delta N$, is related to the change of the interference phase by the equation $\lambda\Delta N = \tau c$, yielding

$$\Delta N = \tau\frac{c}{\lambda} = \frac{v^2}{c^2}\frac{\ell_\perp + \ell_\parallel}{\lambda}. \tag{13}$$

Assuming $v_{Earth} = 30km/s$, and using $\lambda = 6 \times 10^{-7}m$, $\ell = 1.2m$ we have $\Delta N \approx 0.04$. The experimental result was $\Delta N_{obs} = 0 \pm 0.05$. Few years later Michelson and Morley (1887) repeated the experiment be sending the light beam five times across the interferometer, thereby increasing the arms, $\ell \to 10\ell$. The same estimate gives $\Delta N \to 0.4$ as opposed to $\Delta N_{obs} = 0 \pm 0.005$, found.

To explain this null result Fitzgerald and Lorentz came up the proposition in 1892 about a mysterious contraction of solids in motion,

$$\ell \to \ell\sqrt{1-\frac{v^2}{c^2}}, \tag{14}$$

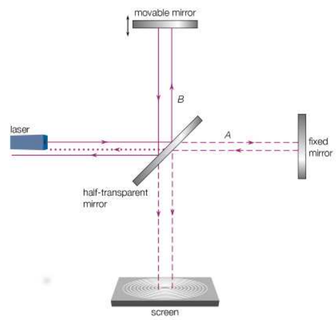



FIG. 9: The schematic sketch of Michelson's experiment.

which yields $\tau = 0$ for $\ell_\perp = \ell_\parallel$. It was difficult to accept that such a universal feature of materials has been unnoticed before and the physicists remained skeptic about this idea until Einstein presented in 1905 a simple analysis of the way length and time are measured in physics which explained the contraction (14). The experimental support of Einstein's view came in 1932 when Kennedy and Thorndike repeated the Michelson-Morley experience with $\ell_\perp \neq \ell_\parallel$ and confirmed the null result.

### E. Special and general relativity

The null result of the Michelson-Morley experiment shows that the addition of the velocity does not apply to light. Hence one of the the following assumptions must be wrong:

1. The physical laws (mechanics and electromagnetism) assume the same form in each reference frame.

2. The time is absolute.

Einstein's special relativity, published in 1905, is based on retaining point 1 and renouncing point 2. The non-absolute, relative nature of time makes the modification of the transformation laws between reference frame necessary. The resulting new transformations, to be discussed below, are named after Lorentz. The new theory asserts that

- The speed of light, being fixed by Maxwell's equations, is the same in each reference frame.

- The coordinate and the velocity can only be given relative to some reference object.

- The acceleration and the derivatives $\frac{d^n \boldsymbol{x}(t)}{dt^n}$, $n \geq 2$ are absolute.

A decade later Einstein came forward with general relativity, where not only the coordinates and the velocities bu all higher order derivatives, $\frac{d^n \boldsymbol{x}(t)}{dt^n}$, are relative.

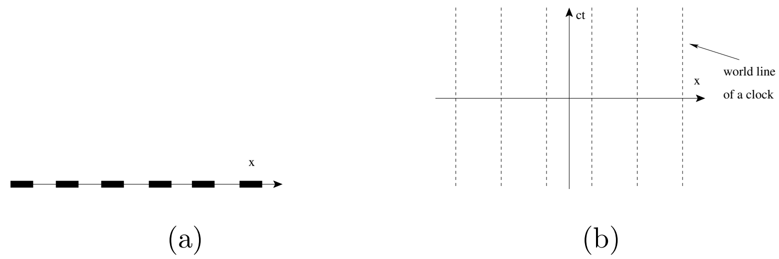



FIG. 10: (a): A spatial coordinate axes. (b): Time coordinate of standard clock.

The understanding of of the need of modifying the transformation (5), connecting reference frames, requires a more careful analysis of the measurement of the coordinates and the time. The coordinates are determined by distances between static points in space, carried out with the desired precision by using a standard, static meter rod. This procedure can be used to construct the coordinate axes with ticks, depicted in Fig. 10 (a), indicating the coordinate values. Next we place a non-moving clock, called standard clock of the coordinate system, at each point in space where we expect an event to take place. Their synchronization will be discussed below. An event is therefore characterized by the answer of two questions, "When?" and "Where?". The respond to the former and the latter consists of one and three numbers, rearranged as the components of a four dimensional vector in the space-time. The set of events, points in space-time, corresponding to a standard clock, showing different time, is the world line of the clock and is indicated in Fig. 10 (b). Note that this procedure, the construction of the coordinate system, is strictly classical, without the possibility of extending it into the quantum domain.

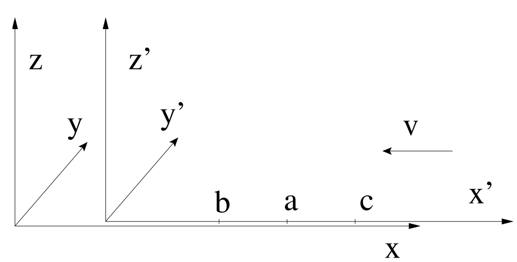



FIG. 11: The light pulse, emitted from the center (a) of a mobile at rest in the reference frame $S$, arrive simultaneously to the end of the mobile, (b( and (c), in $S$. In another reference frame where the mobile moves to the left with velocity $v$ the arrival of the two pulses is not simultaneous anymore.

There is a surprising phenomenon, following from point 1 above, namely the relativity of the simultaneity. Imagine a mobile, having a light source at its center and two light detectors at the ends. The light source produces a short light impulse which arrives at both detectors. In the reference frame $S_0$ where the mobile is at rest the two detectors receive the pulses simultaneously, the detection of the light corresponds to the same time, as measured by the standard clocks of this reference frame. In a reference frame $S'$ where the mobile is moving with a constant velocity $v$ to the left in Fig. 11 one detector moves away (left) and the other (right) towards the space point where the light pulses were emitted. Since the velocity of light is the same the reference frames the former detector receives the light later.

## III. SPACE-TIME

### A. World line

The goal of the Newtonian is the establishment of the trajectory, $\boldsymbol{x}(t)$, three functions of the time. The relativistic dynamics deals with world lines, $x^\mu(s) = (ct(s), \boldsymbol{x}(s))$ with $\mu = 0, 1, 2, 3$, curves in the space times, parametrized by the invariant length of the world line, $s$, to be defined below. The role of the time in the Newtonian mechanics is to distinguish the events and the cause from result, the former taking place earlier the latter when initial conditions are applied. This role is taken over by the parameter $s$ in the relativistic mechanics, leaving the time a new dynamical quantity when compared with the Newtonian case.

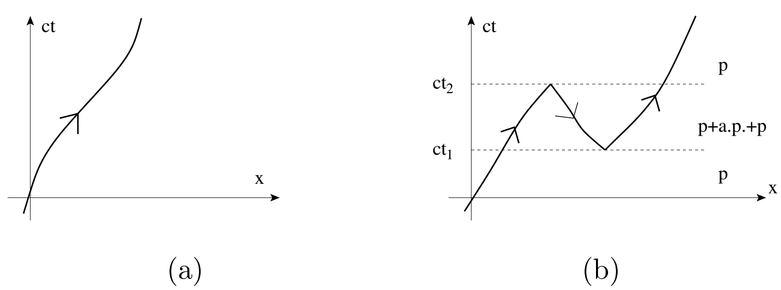
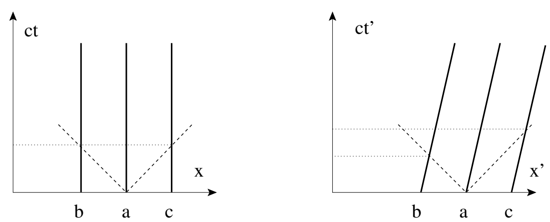



FIG. 12: (a): The world line representation of a Newtonian trajectory (b): A world line without Newtonian analogy.

Any non-relativistic trajectory can be recast in the form of a world line as shown in Fig. 12 (a), the only restriction being $|\dot{x}^0(s)| = |\frac{d}{ds}x^0(s)| > 1$ to keep the particle slower than the light. The new role of time appears in a surprising manner in the existence of world lines without the possibility of representing them by trajectory. This happens if the function $t(s)$ is non-monotonic, a simple example is given by Fig. 12 (b) where to avoid sperluminal motion $\dot{x}^0(s)$ changes sign in a discontinuous manner. The interpretation of this world line is that we a single particle for $t < t_1$ and $t > t_2$ and a two particles and an anti-particle for $t_1 < t < t_2$. Anti-particles are particles whose internal time runs counter clockwise, in the opposite direction than the standard clocks.

The particles anti-particle exchange is called charge conjugation (it changes the sign of the electric charge) and can be represented by the transformation $s \to -s$. or $t \to -t$. This latter is equivalent with $E \to -E$, c.f. the Hamilton equation, $\frac{dp}{dt} = -\frac{\partial H(p,q)}{\partial q}$, $\frac{dq}{dt} = \frac{\partial H(p,q)}{\partial p}$ of classical mechanics and the Schr$\ddot{\text{d}}$inger equation, $i\hbar\partial_t|\psi\rangle = H|\psi\rangle$ in the quantum case.




(a) (b)

FIG. 13: (a): The mobile at rest. (b): The mobile, seen in a moving reference frame.

The space-time is a more appropriate concept to visualize the events in the relativistic case. As an example consider the issue of the simultaneity in Fig. 13. The two light pulse arrive at the same time in the reference frame where the mobile is at rest and obviously define a pair of non-simultaneous events in the moving reference frame.

Finally, let us return to the synchronization of the standard clocks, left open above. We use the clock, located at the origin of the space as the reference, c.f. Fig. 14, and use the following protocol with the observer at the other clocks. (i) The clocks are provided a mirror to reflect lights, received form the reference clock. (ii) A light signal, emitted from the origin at $t = 0$, is reflected back from the clocks and the observers are instructed to note the time, $t(\boldsymbol{x})$, when this takes place. The time $T(\boldsymbol{x})$ of the return of the light from point $\boldsymbol{x}$ to the reference clock is determined, after that the clock at $\boldsymbol{x}$ is shifted by $\Delta t = \frac{T(\boldsymbol{x})}{2} - t(\boldsymbol{x})$.

### B. Lorentz transformation

The Lorentz transformations, $(ct, \boldsymbol{x}) \to (ct', \boldsymbol{x}')$, connect the reference frames. They can not be orthogonal as rotation in Euclidean space, shown in Fig. 15 (a), because they have to

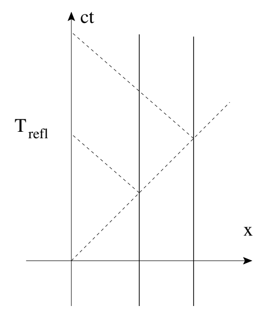



FIG. 14: The synchronization of the standard clocks.

preserve the world line of a light pulse, a straight line with slope $\pm 1$. If we change the coordinate axes, $x \to x'$, then the new time axes must make the same angle with the world line of the light as the space axes. The result, shown in Fig. 15 (b), a coordinate system which is non-orthogonal in the Euclidean sense but remain orthogonal in the relativistic Minkowski geometry. The need of preserving the world line of the light ray in any reference frame introduces fixed lines in the Minkowski space-time with respect to Lorentz transformations.

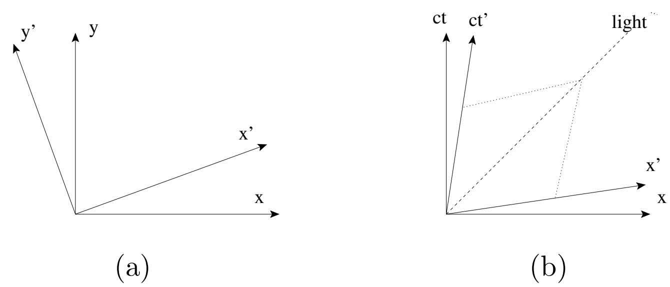



FIG. 15: (a): Rotation in an Euclidean plane. (b): Lorentz boost in a 1+1 dimensional space-time.

The real surprises in special relativity and often in other part of physics are related to the time. The loss of the absolute nature of the simultaneity is maybe the deepest and the least intuitive level a relativity, justifying to present yet another argument in its favor. The events $A$ and $B$, placed on Fig. 16, are simultaneous in the reference frame but $S = (ct, x)$ $A$ is succeeding or preceding $B$ in the reference frames $S = (ct', x')$ or $S = (ct'', x'')$, respectively,

The mathematical expression of the Lorentz boost in a 1+1 dimensional space-time can be found by starting the general expression of the linear transformation of the coordinate, mixing of space and time,

$$x' = ax - bct = a(x - vt), \tag{15}$$

whose inverse, obtained by the change $v \to -v$, is

$$x = a(x' + vt'). \tag{16}$$

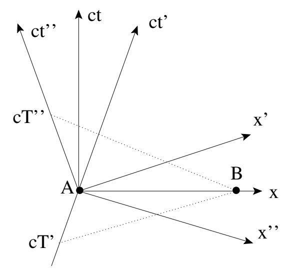



FIG. 16: The relativity of the simultaneity of the events $A$ and $B$.

The application of this transformation for the propagation of the light, $x = ct$, $x' = ct'$,

$$ct' = a(c - v)t, \quad ct = a(c + v)t', \tag{17}$$

gives

$$a = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{18}$$

The transformation of the time can be found by using $x = ct$, resulting the Lorentz boost,

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}, \qquad x = \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}},$$

$$t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}, \qquad t = \frac{t' + \frac{vx'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

which reduces to the Galilean boost in the limit $\frac{v}{c} \to 0$. What is more difficult to prove in such a simple setting is that the distance, in directions orthogonal to the boost velocity remains unchanged.

## C. Addition of the velocity

The Lorentz boost expressions can be used to find out the solution of the addition of velocity paradox. Let us consider two different reference frames, $S$ and $S'$, related by a boost by velocity $u$,

$$x_\| = \frac{x'_\| + ut'}{\sqrt{1 - \frac{u^2}{c^2}}}, \quad \boldsymbol{x}_\perp = \boldsymbol{x}'_\perp, \quad t = \frac{t' + \frac{ux'_\|}{c^2}}{\sqrt{1 - \frac{u^2}{c^2}}} \tag{19}$$

The transformation of the velocity, $\boldsymbol{v} = \frac{d\boldsymbol{x}}{dt} \to \boldsymbol{v}' = \frac{d\boldsymbol{x}'}{dt'}$, can be found by the help of the expressions

$$\Delta x_\| = \frac{(v'_\| + u)\Delta t'}{\sqrt{1 - \frac{u^2}{c^2}}}, \qquad \Delta \boldsymbol{x}_\perp = \boldsymbol{v}'_\perp \Delta t', \qquad \Delta t = \frac{(1 + \frac{uv'_\|}{c^2})\Delta t'}{\sqrt{1 - \frac{u^2}{c^2}}}, \tag{20}$$

in particular,

$$v_\parallel = \frac{\Delta x}{\Delta t} = \frac{v'_\parallel + u}{1+\frac{uv'_\parallel}{c^2}} = \begin{cases} v'_\parallel + u & u, v'_\parallel \ll c \\ c & v'_\parallel \ll c, u \approx c, \quad \text{ou} \quad u \ll c, v'_\parallel \approx c \end{cases}$$

$$\boldsymbol{v}_\perp = \boldsymbol{v}'_\perp \frac{\sqrt{1-\frac{u^2}{c^2}}}{1+\frac{uv'_x}{c^2}} = \begin{cases} v'_y & u \ll c \\ 0 & u \approx c \end{cases}$$

It is instructive to simplify these expression by going into certain limits. In the non-relativistic limit we have $v_\parallel = v'_\parallel + u$ and $\boldsymbol{v}_\perp = \boldsymbol{v}'_\perp$ and the relativistic boost, $u \approx c$ yields $v_\parallel \approx c$ and $\boldsymbol{v}_\perp = 0$. The nonlinearity of the Lorentz boost in the boost velocity keeps the velocity below the speed of light.

### D. Invariant distance

One frequently needs a simple test to decide whether a linear transformation of the space and time can be considered as a transformation between reference frames. A similar problem, the characterization of the linear transformations which preserve the three dimensional spatial Euclidean geometry, rotations, is given by requiring the invariance of the scalar product or the length, $s^2 = (\boldsymbol{x}_2 - \boldsymbol{x}_1)^2$. The generalization of this test to Minkowski geometry is that the symmetry transformations between reference frames preserve the Minkowski invariant length square,

$$s^2 = c^2(t_2 - t_1)^2 - (\boldsymbol{x}_2 - \boldsymbol{x}_1)^2. \tag{21}$$

There are two different proofs of this statement. One is to prove that the Lorentz transformation indeed preserve the length square,

$$s^2 = c^2t^2 - x^2 \to \frac{(ct - \frac{vx}{c})^2 - (x - vt)^2}{1-\frac{v^2}{c^2}} = \frac{(c^2t^2 - x^2)(1-\frac{v^2}{c^2})}{1-\frac{v^2}{c^2}} = c^2t^2 - x^2. \tag{22}$$

Each step in the argument is reversible, hence the invariance is a sufficient and necessary condition. Another, less direct but more physical argument is the following: Note first that $s^2 = 0$, the possibility of exchanging light signal, is a Lorentz invariant statement. The invariance of $s^2 \neq 0$ can be established by the help of three reference frames, $S_0$, and $S_j$, $\boldsymbol{v}_{S_0 \to S_j} = \boldsymbol{v}_j$, $j = 1, 2$, $|\boldsymbol{v}_1|, |\boldsymbol{v}_2| \ll c$. Denote the value of $s^2$ after a Lorentz boost of velocity $\boldsymbol{v}$ by a regular function, $s'^2 = F(|\boldsymbol{v}|, s^2)$. The invariance of the possibility of connecting a pair of events by light, $s^2 = 0$, imposes the condition $F(|\boldsymbol{v}|, 0) = 0$, yielding

$$ds_j^2 = F(|\boldsymbol{v}_j|, ds_0^2) \approx a(|\boldsymbol{v}_j|)ds_0^2, \tag{23}$$

where

$$a(|\boldsymbol{v}|)=\frac{\partial F(|\boldsymbol{v}|,s^2)}{\partial s^2}_{|s^2=0}. \tag{24}$$

There are two ways to obtain the ratio $ds_1^2/ds_2^2$, one is using eq. (23) with $j=1$ and 2 and another is to use the boost $S_1\to S_2$, $ds_2^2=a(|\boldsymbol{v}_1-\boldsymbol{v}_2|)ds_1^2$, resulting the condition

$$a(|\boldsymbol{v}_1-\boldsymbol{v}_2|)=\frac{a(|\boldsymbol{v}_2|)}{a(|\boldsymbol{v}_1|)}. \tag{25}$$

requiring $a=1$.

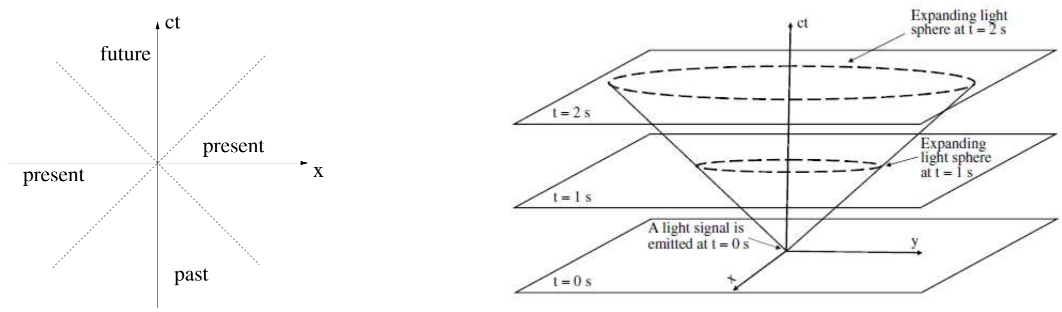
### E. Minkowski geometry

The invariant light rays of the Minkowski geometry induces a particular structure for intervals, defined by pairs of event. The space-time interval of the events $x_1$ and $x_2$ is called

- time-like if $s^2(x_1-x_2)>0$,
- space-like if $s^2(x_1-x_2)<0$,
- light-like if $s^2(x_1-x_2)=0$.

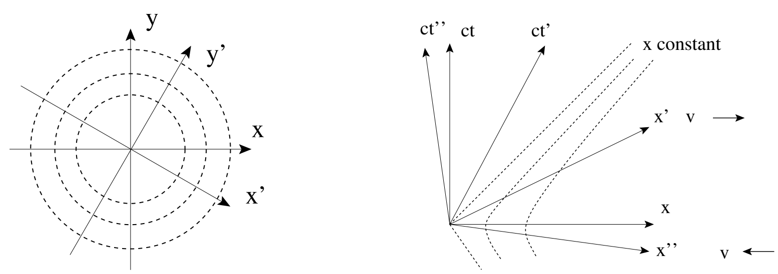
with $s^2(x_1-x_2)=c^2(t_1-t_2)^2-(\boldsymbol{x}_1-\boldsymbol{x}_2)^2$. To find the physical significance of this classification let us start with a 1+1 dimensional space time, shown in Fig. 17 (a). The light signal, emitted along the $t<0$ part of the dashed line, arrive to the origin along light-like intervals. The region, between the dashed lines, consisting of events which are separated from the origin by a time-like interval, is the past since particles, emitted within this region may arrive at the origin. In a similar manner one argues that the region $t>0$ between the dashed lines is the future (of the origin). Any event, lying outside of the two dashed lines, being separated from the origin by a space-like interval, is the present because there are reference frame in which the time of the event is 0. It is remarkable the the present of a point, a single event, is a vast region in the space-time. The generalization of this structure for $t>0$ is shown in Fig. 17 (b) in $1+2$ dimensional space-time, justifying the name light cone of the dashed line of Fig. 17 (a).

An important difference between rotations in Euclidean space and the Lorentz boost in the space-time is the change of scale, suffered during the latter. The rotation of an Euclidean space, shown in Fig. 18 (a), preserves the length scale, the ticks of the coordinate axes make a circle during rotation, $x^2+y^2=R^2$. The image of a an event, in a fixed length square from the origin defines a hyperbole in the space-time in Fig. 18 (b), $(ct)^2-x^2=s^2$, indicating that the invariant length is deformed on the axis.




(a) (b)

FIG. 17: The past, present and future in (a): 1+1 and (b) 1+2 dimensional space-time.




(a) (b)

FIG. 18: (a): Rotation of the Euclidean plane preserves the length scale. (b): The length scale is deformed in a non-linear manner during the Lorentz boosts.

## IV. PHYSICAL PHENOMENAS

The deformation of the scale during the Lorentz transformation generates several important physical effects.

### A. Lorentz contraction of the length

The length of an object is determined from the measurement of the coordinates of the two end points. The non-trivial aspect of this rather natural definition concerns moving objects where the time of the measuring of the end point becomes crucial. It is obvious to demand to perform the measurement simultaneously, within the instantaneously co-moving reference frame.

The proper length of a rod is defined by by measuring simultaneously the difference of the coordinate, $L_0 = x_2 - x_1$, of the end points in the instantaneous co-moving reference frame, $(ct, x)$, as shown in Fig. 19. Let us suppose that this rod is seen moving with a constant velocity in the reference frame $(ct, x')$, then the Lorentz transform of the coordinates of the end points to the co-moving frame,

$$x_a = \frac{x'_a + vt'}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{26}$$

with $a = 1, 2$ yields

$$\ell_0 = \frac{x'_2 - x'_1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{L'}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{27}$$

and the Lorentz contraction,

$$L' = L_0 \sqrt{1 - \frac{v^2}{c^2}}. \tag{28}$$

The contraction is a necessary result of the deformation of the length scale, the hyperboloids, passing the end points in the co-moving frame produces different shifts, $x_a \to x'_a$, for the two end points owing to the non-vanishing slope of the $x'$ axis and the non-linearity of the scale deformation.

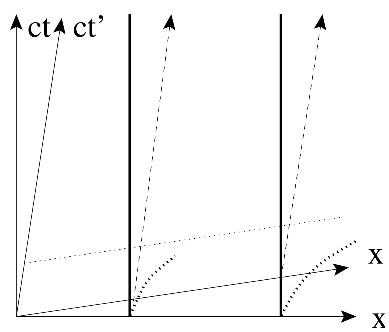



FIG. 19: The determination of the length of a rod in the co-moving and the laboratory reference frames, $(ct, x)$ and $(ct', x')$, respectively.

The apparent contraction of the length, caused by the way the end points of the object are located in a simultaneous manner, can easily be understood by the Lorentz rotation. let us consider a rectangular board of length $L$ and width $W$, indicated by the upper rectangle in Fig. 20, from above. Let us suppose that the board moves to the right with velocity $v$ and observe its motion from the side. The point is that the observation of the end points, $A$ and $B$, is carried out in such a manner that the light, arriving from them reaches our eyes simultaneously, at the same time. Since $A$ is further away from as than $B$ the light leaves $A$ earlier than $B$ hence $A$ must have been left from $B$ at the instant of the emission of the light. Thus we see a rotated board, with end points $A'$ and $B'$. The time delay due to the motion is $\Delta t = \frac{W}{c}$ and the board is moves by $\Delta x = v\Delta t = W\frac{v}{c}$ in the meantime. the angle or the apparent rotation is given by the equation

$$\sin\Theta = \frac{v}{c}, \qquad \cos\Theta = \sqrt{1 - \frac{v^2}{c^2}}, \tag{29}$$

yielding the $W$-independent contraction factor

$$L' = L\sqrt{1 - \frac{v^2}{c^2}}. \tag{30}$$

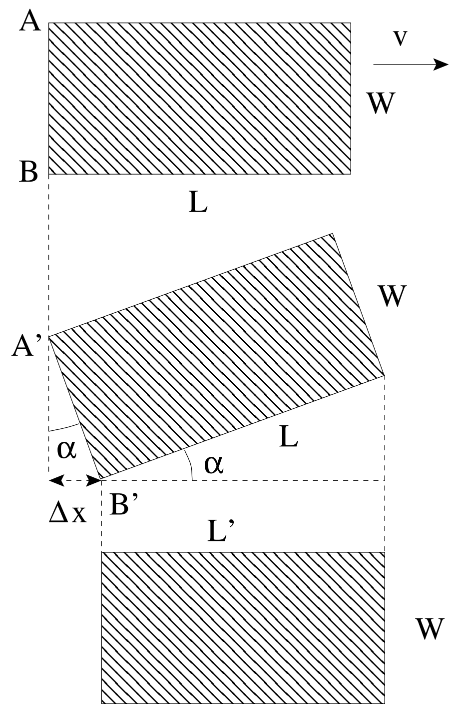



FIG. 20: The apparent rotation of a board, behind the Lorentz contraction of the length.

The contraction of the length is a universal effect, independent of the material composition of the objects, and results from the way the length is observed. Had it been a physical effect the original object, at rest, would have appeared longer. It is easy to see that this is not the case, both the observer, standing beside the object and the other, moving with constant velocity, see shorter length than the other, as shown in Fig. 21.

### B. Time dilatation

The temporal analogy of the Lorentz contraction of the length is called time dilatation. The proper time, $T_0$, measured between two signals of a clock by the standard clock of the instantaneous co-moving reference frame, gives the invariant length square, $s^2 = c^2T_0^2$. Seeing

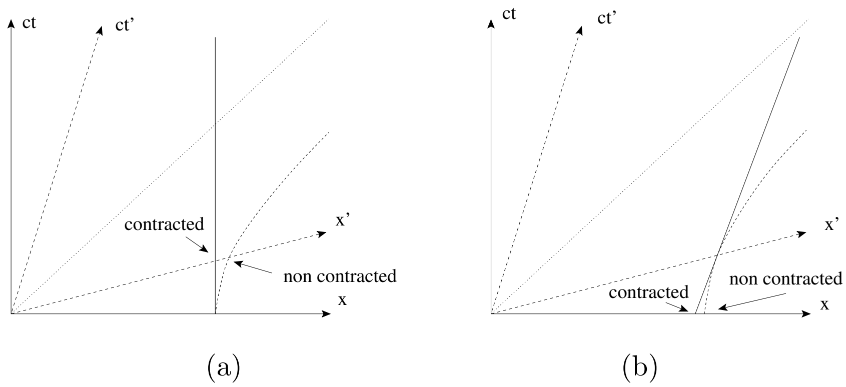



FIG. 21: The Lorentz contraction, seen by the moving observer (a) and an observer at rest with respect to the object.

these events from another reference frame, moving with velocity $v$, we have

$$s^2 = c^2 T'^2 \left(1 - \frac{v^2}{c^2}\right), \tag{31}$$

giving

$$T' = \frac{T_0}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{32}$$

indicating that the moving clocks show shorter time interval, appear to be slower.

The time dilatation is as universal as the Lorentz contraction of the length, and is easy to understand in the case of a particular, idealized optical clock, build on a solid rod by placing light detectors at the two end and letting a light signal be reflected between them. The basic time interval of this clock is $c^2 T_0^2 = L_0^2$, $l_0$ being the proper length of the rod. Let us now move the rod with velocity $v$ in a perpendicular direction. The light signal, shown in Fig. Fig. 22, now traverses a longer distance, $c^2 T'^2 = T'^2 v^2 + T_0^2 c^2$, giving the reduced basic time, (32).

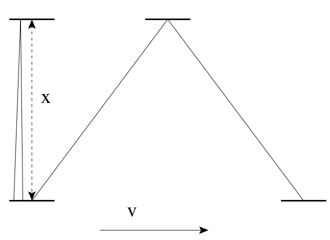



FIG. 22: The time dilation, observed by a moving optical clock.

The optical clock, moving in the direction of the rod can be used to establish the equivalence of the Lorentz contraction of the length and the time dilatation. The time of flight of the light signal from left (a) to right (b) in Fig. 23 and return (c), satisfying the equations $c\Delta T_1 =$

$L_0 + v\Delta T_1$ and $c\Delta T_2 = \ell - v\Delta T_2$, respectively, yields the double basic time

$$2T' = \Delta T_1 + \Delta T_2 = \frac{L'}{c - v} + \frac{L'}{c + v} = \frac{2L'c}{c^2 - v^2} \tag{33}$$

which together with $T_0 c = L_0$ render eqs. (30) and (32) equivalent.

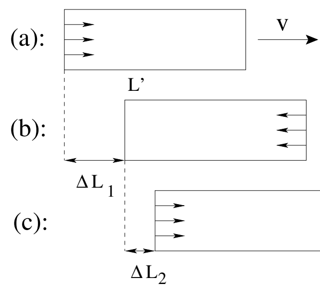



FIG. 23: The optical clock in longitudinal motion. (a): Emission of the light signal at the left end. (b): Emission at the right and. (c): Return of the signal to the left end.

### C. Doppler effect

The Doppler effects, the shift of the received frequency compared with the emitted one for moving sources and observers, is an essential tool for the measurement of the velocity and the temperature of distant objects. The observed shift in the well known discrete emission or absorption spectrum of atoms in the object allows to find out the velocity of the source. The temperature, a measure of the disordered velocity, can be determines from the spread of the spectrum lines. The following four kinds of Doppler effects are known:

**Non-relativistic:** Consider a source and receiver, moving with velocity $u$ and $v$, respectively within a medium where the signal, emitted with frequency $\nu = 1/\Delta t$, propagates with velocity $w$, shown in Fig. 24. During a cycle the wave from makes the distance $\lambda = \delta w = u\Delta t + \lambda'$, $\lambda'$ denoting the wavelength of the signal, emitted form the moving source, hence we have

$$\lambda' = (w - u)\Delta t = \frac{w - u}{\nu} \tag{34}$$

Similar argument, followed at the receiver's end, yields the equation

$$\lambda' = \frac{w - v}{\nu'} \tag{35}$$

where $\nu'$ denotes the received frequency and the relation

$$\nu' = \nu \frac{w - v}{w - u} \tag{36}$$

It is instructive to consider two spacial cases, one is the stationary source, $u = 0$,

$$\nu' = \nu\left(1 - \frac{v}{w}\right), \quad \nu' = 0 \text{ for } v = w \tag{37}$$

the other being the stationary receiver, $v = 0$,

$$\nu' = \frac{\nu}{1 - \frac{u}{w}}, \quad \nu' = \infty \text{ for } u = w. \tag{38}$$

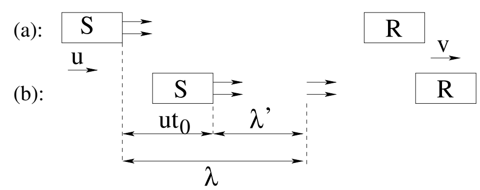



FIG. 24: The non-relativistic Doppler of effect of a source (S) and a receiver (R). The double arrows indicate the location of a wave front, emitted periodically. (a): At the beginning $t = 0$, (b): at the end $t = \Delta t$, of a cycle.

**Relativistic (light):** One follows the emission of two short light pulses with time difference $T$ and the receiver moves with velocity $v$, shown in Fig. 25. The coordinates of the reception of the pulses, $x_1 = ct_1 = x_0 + vt_1$ and $x_2 = c(t_2 - T) = x_0 + vt_2$ yield

$$t_2 - t_1 = \frac{T}{1 - \frac{v}{c}}, \qquad x_2 - x_1 = \frac{vT}{1 - \frac{v}{c}}. \tag{39}$$

The Lorentz boost of the emission time difference into the co-moving frame of the receiver,

$$T' = t_2' - t_1' = \frac{t_2 - t_1 - \frac{v}{c^2}(x_2 - x_1)}{\sqrt{1 - \frac{v^2}{c^2}}} = T\sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}} \tag{40}$$

results the relativistic Doppler effect,

$$\nu' = \nu\sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}}. \tag{41}$$

**Static gravitational field:** A static potential can generate a Doppler effect, as well, and this effect is universal in the case of gravity. Consider an electron-positron pair at rest when they start to fall in homogeneous gravitational field and make a distance $L$, depicted in Fig.26, the initial and the final energy being $E_\uparrow(e^-e^+) = 2mc^2$ and $E_\downarrow(e^-e^+) = E_\uparrow(e^-e^+) + 2m\Delta U$, respectively. The final energy can be expressed in an universal, mass-independent form,

$$E_\downarrow(e^-e^+) = E_\uparrow(e^-e^+)\left(1 + \frac{\Delta U}{c^2}\right). \tag{42}$$

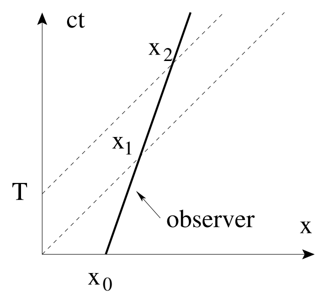



FIG. 25: The non-relativistic Doppler of effect of a source (S) and a receiver (R). The double arrows indicate the location of a wave front, emitted periodically. (a): At the beginning $t = 0$, (b): at the end $t = \Delta t$, of a cycle.

Let us assume that the electron-positron pair annihilates, $E_\downarrow(e^-e^+) = \hbar\omega_\downarrow$, and the photon propagates back to the initial point of the pair. The photon energy at that point must be the same as the initial energy of the pair, $E_\uparrow(e^-e^+) = \hbar\omega_\uparrow$ hence

$$\frac{\omega_\downarrow}{\omega_\uparrow} = \frac{E_\downarrow(e^-e^+)}{E_\uparrow(e^-e^+)} = 1 + \frac{\Delta U}{c^2}. \tag{43}$$

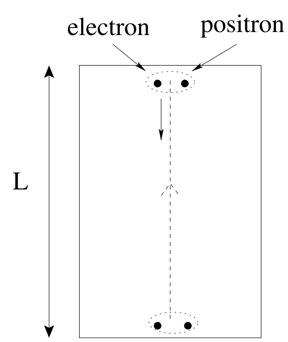



FIG. 26: The non-relativistic Doppler of effect of a source (S) and a receiver (R). The double arrows indicate the location of a wave front, emitted periodically. (a): At the beginning $t = 0$, (b): at the end $t = \Delta t$, of a cycle.

**Time-dependent gravitational field:** A time-dependent potential generates Doppler effect in an obvious manner by modifying the frequencies and the wavelengths. We consider the universal effect of a time-dependent gravitational field of the Robertson-Walker model of the Universe, predicting that the observed wavelength is longer than the emitted one, $\lambda_{em} < \lambda_{obs}$ due to the expansion of the Universe, defining the red shift,

$$\frac{\lambda_{obs}}{\lambda_{em}} = \frac{\omega_{em}}{\omega_{obs}} = 1 + z. \tag{44}$$

The Hubble law states that the red shift is proportional with the distance,

$$z = \frac{H}{c}\ell, \tag{45}$$

where $H$ denotes the Hubble constant.

## D. Paradoxes

The obviously contradicting features of the Newtonian and the relativistic mechanics lead to more subtle relativistic "paradoxes", surprising phenomenons.

**Twins:** There are two twins, having the same age. One of them is taken to the Moon and back with a space ship, the other stays on the Earth. Which one is older than the other when they meet again? Both may argue that he saw the other moving away and back hence the other is younger. Who is right?

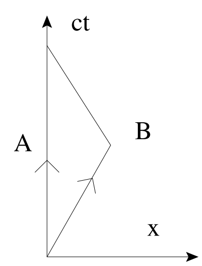



FIG. 27: The world lines of the wins A and B.

*Solution:* Let us accept the simplification, treating the Earth as a reference frame. The symmetry between the two argument is broken by realizing that the person traveling had to undergone an acceleration when the space ship turned back. Hence he does not follow the motion of a reference frame and his argument is incorrect, making him the younger.

But there is still an interesting problem, hiding in this paradox, the amount of age difference. if it is due to the acceleration at turning back then it should be independent of the time spent with the travel. The detailed calculation, similar to the way the relativistic Doppler effect is handled, yields a n age difference, proportional to the travel time.

**A rod and a circle:** We have a rod and a sheet, with a circle cut out in such a manner that the length of the rod of negligible width is the diameter of the circle with negligible thickness. They move with constant velocity and approach each other as shown in Fig. 28, meeting in such a manner that their centers coincide. Can the rod pass through the circle? The paradox is the following: On the one hand, seeing from the reference frame of the rod we see a Lorentz contracted circle approaching hence we expect the rod not to get through. On the other hand, seeing from the reference frame of the circle the contracted rod can safely pass.

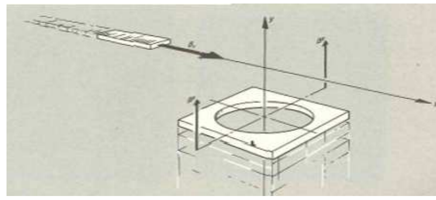
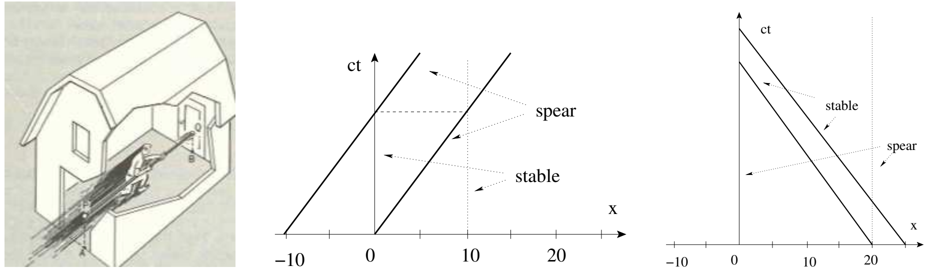


FIG. 28: The rod and the circle.

*Solution:* The circle suffers a Lorentz rotation, discussed above, making the crossing possible.

**A spear and a stable:** there is a person, running with a spear of length $\ell$ with a constant speed $v$, satisfying the equation $\sqrt{1-\frac{v^2}{c^2}}=\frac{1}{2}$, towards a stable of length $\ell/2$. Is there an instant when the person and the spear are completely within the stable?




(a) (b) (c)

FIG. 29: (a): The spear and the stable. The arrival to the stable, seen in the reference frame of the stable (b) and the spear (c).

The arrival to the stable, seen from the reference frame of the stable, appears as the spear of the same length than the stable approaches hence there will be an instant when spear is within the stable. However, seen from the reference frame of the person, one finds that the length of the stable is the quarter of the spear hence there is no way of fitting in.

*Solution:* The idea "be within the stable" requires that both end of the spear are simultaneously inside, within the reference frame, fixed to the stable. This condition is satisfied.

# V. ELEMENTS OF RELATIVISTIC MECHANICS

## A. Vectors and tensors

The events in the space-time are identified by the help of the four-vector coordinate, $x^\mu = (ct, \boldsymbol{x}) = (x^0, \boldsymbol{x})$, $\mu = 0, 1, 2, 3$, where the time is expressed in unit of length. The components are mixed by the Lorentz transformations, characterized by the invariance of the length square,

$$s^2 = x^{02} - \boldsymbol{x}^2 = \sum_{\mu\nu} x^\mu g_{\mu\nu} x^\nu = x \cdot x, \tag{46}$$

where

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{47}$$

is the metric tensor. It defines the scalar product,

$$x \cdot y = \sum_{\mu\nu} x^\mu g_{\mu\nu} y^\nu \tag{48}$$

where we write in a simpler form where the metric tensor is hidden by using two forms of the vectors. The covariant, $u_\mu$, and the contravariant, $u^\mu$, versions are related by the metric tensor,

$$u_\mu = \sum_\nu g_{\mu\nu} u^\nu, \quad u^\mu = \sum_\nu g^{\mu\nu} u_\nu, \tag{49}$$

making possible to write the scalar product as

$$x \cdot y = x^\mu y_\mu = x_\mu y^\mu \tag{50}$$

where the Einstein convention is employed, namely the summation is not shown to simplify the expressions. Tensors of rank $n$ have $2^n$ versions, each index being either covariant or contravariant. The consistency of the two covariant and the contravariant versions requires

$$g_{\mu\nu} g^{\nu\rho} = g_{\mu}{}^{\cdot}{}_{\cdot}{}^{\rho} = \delta_\mu^\rho \tag{51}$$

The Lorentz transformations,

$$x^\mu \to x'^\mu = \Lambda^\mu{}_{\cdot}{}^{\cdot}{}_\nu x^\nu, \tag{52}$$

preserve the invariant length,

$$xy = x^{\mu'} \Lambda^\mu{}_{\cdot}{}^{\cdot}{}_{\mu'} g_{\mu\nu} \Lambda^\nu{}_{\cdot}{}^{\cdot}{}_{\nu'} y^{\nu'}. \tag{53}$$

Hence the Lorentz group consists of real $4 \times 4$ matrices, satisfying the condition

$$g_{\mu\nu} = \Lambda^{\mu'}_{\cdot\ \mu}{}^{\cdot} g_{\mu'\nu'} \Lambda^{\nu'}_{\cdot\ \nu}{}^{\cdot}. \tag{54}$$

It is a continuous group if 6 dimensions, 3 for spatial rotations,

$$\Lambda = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & R \end{pmatrix}, \quad R^{\mathrm{tr}} R = \mathbb{1}, \tag{55}$$

and another 3 dimensions for Lorentz boosts. The latter can be found by using the decomposition $\boldsymbol{x} = \boldsymbol{x}_\parallel + \boldsymbol{x}_\perp$ with $\boldsymbol{x}_\parallel \boldsymbol{x}_\perp = \boldsymbol{v}\boldsymbol{x}_\perp = 0$ and the parametrization

$$\boldsymbol{x}' = \alpha(\boldsymbol{x}_\parallel - \boldsymbol{v}t) + \gamma \boldsymbol{x}_\perp, \quad t' = \beta\left(t - \frac{\boldsymbol{x}_\parallel \boldsymbol{v}}{\tilde{c}^2}\right) \tag{56}$$

of the transformation. The invariance of the length square (46),

$$\begin{aligned} c^2t^2 - \boldsymbol{x}^2 &= c^2\beta^2\left(t - \frac{\boldsymbol{x}_\parallel \boldsymbol{v}}{\tilde{c}^2}\right)^2 - \alpha^2(\boldsymbol{x}_\parallel - \boldsymbol{v}t)^2 - \gamma \boldsymbol{x}_\perp^2 \\ &= c^2t^2\left(\beta^2 - \alpha^2\frac{\boldsymbol{v}^2}{c^2}\right) - \boldsymbol{x}_\parallel^2\left(\alpha^2 - \frac{c^2}{\tilde{c}^2}\frac{\boldsymbol{v}^2}{\tilde{c}^2}\beta^2\right) + 2\boldsymbol{v}\boldsymbol{x}_\parallel t\left(\alpha^2 - \frac{c^2}{\tilde{c}^2}\beta^2\right) - \gamma^2 \boldsymbol{x}_\perp^2 \end{aligned}$$

yeilds the conditions

$$\begin{aligned} \mathcal{O}\left(\boldsymbol{x}_\perp^2\right): \ \gamma &= \pm 1 \to 1 \quad \leftarrow \quad v = 0 \\ \mathcal{O}\left(\boldsymbol{x}_\perp \boldsymbol{v}\right): \ 0 &= \alpha^2 - \frac{c^2}{\tilde{c}^2}\beta^2 \\ \mathcal{O}\left(\boldsymbol{x}_\parallel^2\right): \ -1 &= \frac{c^2}{\tilde{c}^2}\frac{\boldsymbol{v}^2}{\tilde{c}^2}\beta^2 - \alpha^2 \ \to \ \beta^2 = \frac{\tilde{c}^2}{c^2}\frac{1}{1 - \frac{\boldsymbol{v}^2}{\tilde{c}^2}} \\ \mathcal{O}\left(c^2t^2\right): \ 1 &= \beta^2 - \alpha^2\frac{\boldsymbol{v}^2}{c^2} = \frac{\tilde{c}^2}{c^2}\frac{1}{1 - \frac{c^2}{\tilde{c}^2}\frac{\boldsymbol{v}^2}{c^2}}\left(1 - \frac{c^2}{\tilde{c}^2}\frac{\boldsymbol{v}^2}{c^2}\right) \ \to \ \tilde{c}^2 = c^2, \end{aligned}$$

where the sign in the first eqaution was chosen to assure the trivial transformation for $\boldsymbol{v} = 0$, i.e. to exclude spatial inversion, a particular Lorentz transformation. The result is

$$\alpha = \beta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{57}$$

after having chosen the positive root by the same argument. We have finally

$$\boldsymbol{x}'_\parallel = \frac{\boldsymbol{x}_\parallel - \boldsymbol{v}t}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad ct' = \frac{ct - \frac{\boldsymbol{v}\boldsymbol{x}_\parallel}{c}}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{58}$$

and the inverse transformation is found by the change $\boldsymbol{v} \to -\boldsymbol{v}$,

$$\boldsymbol{x}_\parallel = \frac{\boldsymbol{x}'_\parallel + \boldsymbol{v}t'}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad ct = \frac{ct' + \frac{\boldsymbol{v}\boldsymbol{x}'_\parallel}{c}}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{59}$$

**Exercise:** To check the internal consistency of the formalism prove the following equations:

$$\begin{aligned}
g &= \Lambda^{\rm tr} g \Lambda \\
\Lambda^{-1} &= g^{-1}\Lambda^{\rm tr} g = (g\Lambda g^{-1})^{\rm tr}, \qquad \leftrightarrow \qquad \Lambda^{-1\mu}{}_{\nu} = g^{\mu\mu'}\Lambda^{\nu'}{}_{\mu'} g_{\nu'\nu} = \Lambda_{\nu}{}^{\mu} \\
x'^{\mu} &= (\Lambda x)^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \\
x^{\mu} &= (g\Lambda g^{-1})_{\nu}{}^{\mu} x'^{\nu} = x'^{\nu}\Lambda_{\nu}{}^{\mu} = (x'\Lambda)^{\mu} \\
x'_{\mu} &= (g\Lambda x)_{\mu} = (g\Lambda g^{-1} g x)_{\mu} = \Lambda_{\mu}{}^{\nu} x_{\nu} = (\Lambda x)_{\mu} \\
x_{\mu} &= x'^{\nu} g_{\nu\lambda} g^{\lambda\rho}\Lambda_{\rho\mu} = x'_{\lambda}\Lambda^{\lambda}{}_{\mu} = (x'\Lambda)_{\mu}
\end{aligned}$$

where the notation $\Lambda = \Lambda^{\mu}{}_{\nu}$, $g = g_{\mu\nu}$ and $g^{\rm tr} = g$ is used for the condensed equations, shown without indices. Note the similarity of these equations with the orthogonality condition $R^{\rm tr}R = \mathbb{1}$ and $\boldsymbol{x}' = R\boldsymbol{x}$, $\boldsymbol{x} = R^{\rm tr}\boldsymbol{x}' = \boldsymbol{x}'R$, holding for Euclidean rotations.

The tensors are defined by applying the Lorentz transformation on the indices independently,

$$\begin{aligned}
\text{contravariant} &: T^{\mu_1\cdots\mu_n} = \Lambda^{\mu_1}{}_{\nu_1}\cdots\Lambda^{\mu_n}{}_{\nu_n} T^{\nu_1\cdots\nu_n} \\
\text{covariant} &: T_{\mu_1\cdots\mu_n} = \Lambda_{\mu_1}{}^{\nu_1}\cdots\Lambda_{\mu_n}{}^{\nu_n} T_{\nu_1\cdots\nu_n} \\
\text{mixed} &: T^{\rho_1\cdots\rho_m}_{\mu_1\cdots\mu_n} = \Lambda^{\rho_1}{}_{\kappa_1}\cdots\Lambda^{\rho_m}{}_{\kappa_m}\Lambda_{\mu_1}{}^{\nu_1}\cdots\Lambda_{\mu_n}{}^{\nu_n} T^{\kappa_1\cdots\kappa_m}_{\nu_1\cdots\nu_n}
\end{aligned}$$

There are two invariant tensors:

1. The definition (54) of the Lorentz group implies that the metric tensor remains the same in each reference frame, $g_{\mu\nu} = \Lambda^{\mu'}{}_{\mu} g'_{\mu'\nu'}\Lambda^{\nu'}{}_{\nu}$. Naturally the same holds for $g^{\mu\nu}$ and $g^{\nu}_{\mu}$.

2. The antisymmetric tensor, $\epsilon^{\mu\nu\rho\sigma} = 0, \pm 1$, $\epsilon^{\mu\nu\rho\sigma} = -\epsilon^{\nu\mu\rho\sigma} = -\epsilon^{\mu\rho\nu\sigma} = -\epsilon^{\mu\nu\sigma\rho}$ satisfies the equation
$$\Lambda^{\mu}{}_{\mu'}\Lambda^{\nu}{}_{\nu'}\Lambda^{\rho}{}_{\rho'}\Lambda^{\sigma}{}_{\sigma'}\epsilon^{\mu'\nu'\rho'\sigma'} = \epsilon^{\mu\nu\rho\sigma}\det\Lambda, \tag{60}$$
hence $\epsilon^{\mu\nu\rho\sigma}$ is a pseudo tensor, a tensor which changes sign under space or time inversion.

### B. Relativistic generalization of the Newtonian mechanics

The three-vectors are generalized to four vectors in the relativistic treatment. The four-velocity, the derivative of the world line with respect to the invariant length, is calculated by the help of the relation $ds = dx^0\sqrt{1-\frac{v^2}{c^2}}$,

$$u^{\mu} = \frac{dx^{\mu}(s)}{ds} = \dot{x}(s) = \left(\frac{dx^0}{ds}, \frac{dx^0}{ds}\frac{\boldsymbol{v}}{c}\right) = \left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}, \frac{\frac{\boldsymbol{v}}{c}}{\sqrt{1-\frac{v^2}{c^2}}}\right). \tag{61}$$

The four-acceleration, defined by

$$\dot{u}^\mu = \frac{du^\mu}{ds}, \tag{62}$$

satisfies the equations $u^2 = 1$ and its derivative with respect to $s$, $\dot{u}u = 0$. The four-momentum is defined as

$$p^\mu = mcu^\mu = \left(\frac{mc}{\sqrt{1-\frac{v^2}{c^2}}}, \frac{m\boldsymbol{v}}{\sqrt{1-\frac{v^2}{c^2}}}\right) = \left(\frac{E}{c}, \boldsymbol{p}\right), \tag{63}$$

and it satisfies the mass-shell condition,

$$p^2 = m^2c^2. \tag{64}$$

Finally the four-force is defined by the equation of motion,

$$\frac{dp^\mu}{ds} = \frac{d}{ds}\left(mc\frac{dx^\mu}{ds}\right) = K^\mu. \tag{65}$$

One expects three equations of motion and the temporal components of the equation of motion is beyond the relativistic generalization of Newton's equation. The spatial components of (65),

$$\frac{dt}{ds}\frac{d}{dt}\left(mc\frac{dt}{ds}\frac{d\boldsymbol{x}}{dt}\right) = \boldsymbol{K} \tag{66}$$

can be written in the form

$$\frac{d}{dt}\left(m(v)\frac{d\boldsymbol{x}}{dt}\right) = \frac{d}{dt}\boldsymbol{p} = \boldsymbol{F} \tag{67}$$

by the help of the velocity-dependent mass,

$$m(v) = m\frac{dx^0}{ds} = \frac{m}{\sqrt{1-\frac{v^2}{c^2}}}, \tag{68}$$

and the three-force $\boldsymbol{F} = \frac{ds}{dt}\boldsymbol{K}$. Note that the correct Newtonian limit is recovered as $c \to \infty$ and $\boldsymbol{p} \to \infty$ when $v \to c$. The temporal component,

$$\frac{d}{ds}\left(mc\frac{dx^0}{ds}\right) = \frac{d}{ds}\frac{mc}{\sqrt{1-\frac{v^2}{c^2}}} = K^0 \tag{69}$$

and the orthogonality $Ku = 0$ give

$$\underbrace{K^0}_{\frac{dt}{ds}\frac{d}{dt}cm(v)}\frac{dx^0}{ds} = \underbrace{\boldsymbol{K}}_{\frac{dt}{ds}\boldsymbol{F}}\underbrace{\boldsymbol{u}}_{\frac{dt}{ds}\boldsymbol{v}} \tag{70}$$

which can be recast into the form

$$\frac{dm(v)c^2}{dt} = \boldsymbol{F}\boldsymbol{v}. \tag{71}$$

Hence the temporal component of the equation of motion is energy equation, describing the rate of change of the energy

$$m(v)c^2 = \int \boldsymbol{F}\boldsymbol{v}dt = \int \boldsymbol{F}d\boldsymbol{r}. \tag{72}$$

In the case of a conservative force, $\boldsymbol{F} = -\boldsymbol{\nabla}U$, the energy $E = m(v)c^2 + U = cp^0 + U$ is conserved. Therefore the four momentum can be written into the form

$$p^\mu = \left(\frac{E}{c}, m(v)\boldsymbol{v}\right), \tag{73}$$

and the mass-shell condition gives

$$\frac{E^2}{c^2} = \boldsymbol{p}^2 + m^2c^2, \tag{74}$$

and

$$E(\boldsymbol{p}) = \pm c\sqrt{\boldsymbol{p}^2 + m^2c^2}. \tag{75}$$

Note that the group velocity,

$$\frac{\partial|E(\boldsymbol{p})|}{\partial\boldsymbol{p}} = \frac{\boldsymbol{p}c}{\sqrt{\boldsymbol{p}^2 + m^2c^2}} = \frac{\frac{\boldsymbol{p}}{m}}{\sqrt{1 + \frac{\boldsymbol{p}^2}{m^2c^2}}} = \frac{\frac{\boldsymbol{v}}{\sqrt{1-\frac{\boldsymbol{v}^2}{c^2}}}}{\sqrt{1 + \frac{\boldsymbol{v}^2}{c^2(1-\frac{\boldsymbol{v}^2}{c^2})}}} = \boldsymbol{v}, \tag{76}$$

is bounded, $|\boldsymbol{v}| \leq c$.

**Exercise 1:** The simplest mechanical problem is about the motion of a particle under the influence of a constant force. Let us first assume for the sake of simplicity that the three vector

$$F = \dot{\boldsymbol{p}} = \frac{d}{dt}\frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{77}$$

is constant. We shall use the initial conditions $x_i = v_i = 0$, giving

$$Ft = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{78}$$

and

$$v = \frac{c}{\sqrt{1 + \left(\frac{mc}{Ft}\right)^2}} \approx \begin{cases} t\frac{F}{m} & Ft \ll mc, \\ c & Ft \gg mc. \end{cases} \tag{79}$$

The qualitative feature of the solution is depicted in Fig. 30.

**Exercise 2:** Let us now assume that all components of the force are constant in the instantaneous co-moving frame of the particle where $u^\mu_0 = (1, 0)$, $\dot{u}^\mu_0 = (0, \frac{a}{c^2})$, and $\dot{u}^2 = -\frac{a^2}{c^4}$. The solution, corresponding to the initial conditions $x_i = v_i = 0$, is a hyperboloid,

$$u^\mu = \left(\cosh\frac{a}{c^2}s, \sinh\frac{a}{c^2}s\right),$$

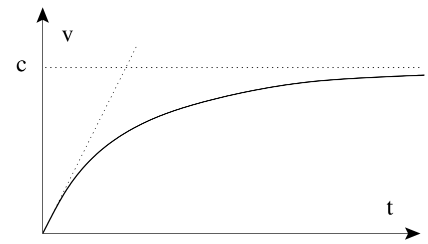



FIG. 30: The velocity as the function of the invariant length in the case of a constant three-force.

$$\dot{u}^\mu = \frac{a}{c^2}\left(\sinh\frac{a}{c^2}s, \cosh\frac{a}{c^2}s\right),$$
$$x^\mu = \frac{c^2}{a}\left(\sinh\frac{a}{c^2}s, \cosh\frac{a}{c^2}s - 1\right),$$

shown in Fig. 31.

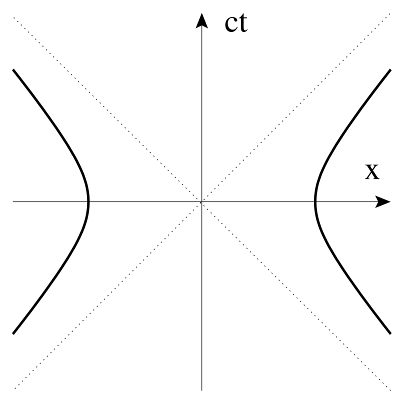



FIG. 31: The world line with constant acceleration in the co-moving frame.

### C. Interactions

The relativistic symmetries of the equation of motion brings a radical change in the strategy of dealing with interactive systems. A system of $N$ non-relativistic particle is usually described by a set of equations of motion,

$$\frac{d^2\boldsymbol{x}_a}{dt^2} = \boldsymbol{F}_a(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n), \tag{80}$$

$a = 1, \ldots, N$, used together with the initial conditions $\boldsymbol{x}_a(t_i) = \boldsymbol{x}_{ia}$, $\frac{d\boldsymbol{x}_a(t_i)}{dt} = \boldsymbol{v}_{ia}$. The relativistic extension of these equations,

$$\ddot{x}^\mu_a = F^\mu_a(x_1, \ldots, x_n), \tag{81}$$

and $x_a(s_i) = x_{ia}$, $\dot{x}_a(s_i) = u_a$, is incompatible with relativistic symmetries. The reason is the direct interaction, described by the four-force, $F^\mu_a$, excluding retardation and generating superluminal effects. A more formal way of locating the problem is to notice that a Lorentz boost, shown in Fig. 32, changes the time axes and couples the initial conditions and the solution of the equation of motion. The kinematic constraint, $\dot{x}\ddot{x} = 0$, requires the orthogonality of the initial velocity and the force, $u_a F_a = 0$.

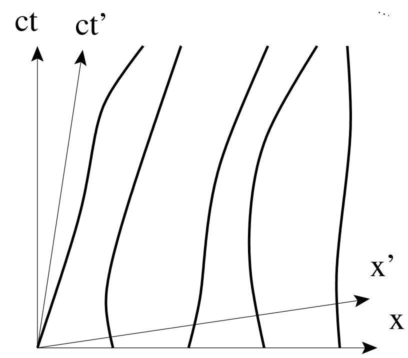



FIG. 32: A Lorentz boost couples the initial conditions and the solution of the equation of motion.

This issue has come to an end by a No-Go theorem, stating that there is no relativistic interaction, mediated by a force of the type (81) which satisfies $u_a F_a = 0$, e.g. the choice

$$F^\mu_a(x_1, \ldots, x_n) = \sum_{b \neq a} (x^\mu_a - x^\mu_b) f((x_a - x_b)^2) \tag{82}$$

would impose the restriction $(x_a - x_b)\dot{x}_a \neq 0$ on the initial conditions.

The solution of this problem is relativistic field theory, the distribution of new dynamical degrees of freedom $\phi(\boldsymbol{x})$, called field, at each space point in such a manner that they couple locally to the matter particles to the other field variable in their vicinity in such a manner that the modification of the world line of a particle generates a retarded response which propagates to the other particle in a manner, compatible with relativistic symmetries.

## VI. VARIATIONAL FORMALISM OF A POINT PARTICLE

The historical origin of the variation principle is the need of writing the equation of motion in a coordinate-independent manner, which preserves its form under arbitrary non-linear, time dependent coordinate transformation.

### A. A point on a line

The toy problem is the identification of a point, $x_{cl} \in \mathbb{R}$, in a way which remains invariant under the reparametrization of the real axes. It is achieved by the help of a function, $S(x)$, which has a only single extreme, at $x_0$,

$$\frac{dS(x)}{dx}_{|x=x_{cl}} = 0. \tag{83}$$

During the reparametrization of the line, $x \to y(x)$ with finite $y'(x)$, this equation transforms into

$$\frac{dS(x(y))}{dy}_{|y=y_{cl}} = \frac{dS(x)}{dx}_{|x=x_{cl}} \frac{dx(y)}{dy}_{|y=y_{cl}} = 0, \tag{84}$$

an equation which is equivalent with (83).

The variational principle, an alternative way to write (83) is to perform a variation, $x \to x + \delta x$, to calculate

$$S(x_{cl} + \delta x) = S(x_{cl}) + \delta x S'(x_{cl}) + \frac{\delta x^2}{2} S''(x_{cl}) + \mathcal{O}\left(\delta x^3\right) = S(x_{cl}) + \delta S(x_{cl}) \tag{85}$$

and to require

$$\delta S(x_{cl}) = \mathcal{O}\left(\delta x^2\right). \tag{86}$$

This equation, being written without the use of a particular coordinate system, is independent of the parametrization of the real axes.

### B. Non-relativistic particle

The extension of the previous construction for a trajectory, $x_{cl}(t)$, satisfying the equation of motion with the auxiliary conditions $x_{cl}(t_i) = x_i$, $x_{cl}(t_f) = x_f$, starts with performing the variation, $x(t) \to x(t) + \delta x(t)$, with $\delta x(t_i) = \delta x(t_f) = 0$, of the action

$$S[x] = \int_{t_i}^{t_f} dt L(x(t), \dot{x}(t)), \tag{87}$$

a local functional of the trajectory, parametrized by the Lagrangian, $L(x(t), \dot{x}(t))$ where $\dot{x} = \frac{dx}{dt}$ in this section about non-relativistic mechanics. The variational equation,

$$\delta S[x] = \mathcal{O}\left(\delta^2 x\right), \tag{88}$$

for

$$\delta S[x] = \int_{t_i}^{t_f} dt L\left(x(t) + \delta x(t), \dot{x}(t) + \frac{d}{dt}\delta x(t)\right) - \int_{t_i}^{t_f} dt L(x(t), \dot{x}(t))$$

$$
\begin{aligned}
&= \int_{t_i}^{t_f} dt \Big[ L(x(t),\dot{x}(t)) + \delta x(t) \frac{\delta L(x(t),\dot{x}(t))}{\delta x} \\
&\quad + \frac{d}{dt}\delta x(t) \frac{\delta L(x(t),\dot{x}(t))}{\delta \dot{x}} + \mathcal{O}\left(\delta x(t)^2\right) - \int_{t_i}^{t_f} dt L(x(t),\dot{x}(t)) \Big] \\
&= \int_{t_i}^{t_f} dt \delta x(t) \left[ \frac{\delta L(x(t),\dot{x}(t))}{\delta x} - \frac{d}{dt} \frac{\delta L(x(t),\dot{x}(t))}{\delta \dot{x}} \right] \\
&\quad + \underbrace{\delta x(t)}_{0} \frac{\delta L(x(t),\dot{x}(t))}{\delta \dot{x}} \bigg|_{t_f}^{t_i} + \mathcal{O}\left(\delta x(t)^2\right)
\end{aligned}
$$

leads to the Euler-Lagrange equation,

$$
\frac{d}{dt} \frac{\partial L(x,\dot{x})}{\partial \dot{x}} = \frac{\partial L(x,\dot{x})}{\partial x}. \tag{89}
$$

The Lagrangian,

$$
L = T - U = \frac{m}{2} \dot{\boldsymbol{x}}^2 - U(\boldsymbol{x}), \tag{90}
$$

generates

$$
m\ddot{\boldsymbol{x}} = -\boldsymbol{\nabla} U(\boldsymbol{x}). \tag{91}
$$

It is a non-trivial exercise to show that (89) preserves its form under non-linear, time-dependent coordinate transformations. A word of caution, the choice $L = T - U$ is made only in the simple mechanical cases, a more involved Lagrangian may be needed in field theory.

The comparison of (89) with Newton's equation of motion suggests the call

$$
p = \frac{\delta L}{\delta \dot{x}} \tag{92}
$$

the generalized momentum, corresponding to the coordinate $x$. The coordinate $x$ is called cyclic if it is not present in the Lagrangian. The generalized momentum of a cyclic coordinate is a conserved quantity

$$
\dot{p}_{cycl} = \frac{\partial L(x,\dot{x})}{\partial x_{cycl}} = 0 \tag{93}
$$

according to the Euler-Lagrange equation (89).

The Hamiltonian, the energy, expressed in terms of the coordinate and the momentum, is obtained by a Legendre transformation,

$$
H(p,x) = p\dot{x} - L(x,\dot{x}), \quad p = \frac{\delta L}{\delta \dot{x}}. \tag{94}
$$

The energy is conserved for time independent Lagrangian,

$$
\dot{H} = \dot{p}\dot{x} + p\ddot{x} - \dot{x}\frac{\delta L}{\delta x} - \ddot{x}\frac{\delta L}{\delta \dot{x}} = 0, \tag{95}
$$

and the equation of motion in the phase space, $(x, p)$,

$$\dot{x} = \frac{\partial H}{\partial p},$$
$$\dot{p} = -\frac{\partial H}{\partial x},$$

is called Hamilton equation.

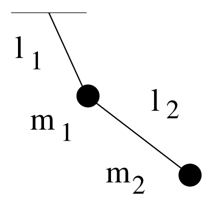



FIG. 33: The double pendulum.

Few examples are in order at this point to demonstrate the simplicity of the derivation of the equations of motion by the help of the variational prciniple.

**Example 1:** Let us consider a point particle of mass $m$ moving on the curve, $y = f(x)$, on the $(x, y)$ plane: $\dot{y} = f'(x)\dot{x}$ under the influence of a homogeneous gravitational potential. The rule $L = T - U$ yields

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - mgf(x) = \frac{m(x)}{2}\dot{x}^2 - mgf(x), \tag{96}$$

where $m(x) = m[1 + f'^2(x)]$ is a space-dependent effective mass.

**Example 2:** The Lagrangian of a pendulum is

$$L = \frac{m}{2}\ell^2\dot{\theta}^2 + mg\ell\cos\theta \tag{97}$$

**Example 3:** The double pendulum consists of two point particles as shown in Fig. 33. The position of the particles can be parametrized by the angles $\theta_1$, $\theta_2$,

$$\boldsymbol{x}_1 = \ell_1 \begin{pmatrix} \sin\theta_1 \\ -\cos\theta_1 \end{pmatrix}, \quad \boldsymbol{x}_2 = \boldsymbol{x}_1 + \ell_1 \begin{pmatrix} \sin\theta_2 \\ -\cos\theta_2 \end{pmatrix}$$
$$\dot{\boldsymbol{x}}_1 = \ell_1\dot{\theta}_1 \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix}, \quad \dot{\boldsymbol{x}}_2 = \dot{\boldsymbol{x}}_1 + \ell_2\dot{\theta}_2 \begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix}$$

and the Lagrangian,

$$L = \frac{m_1 + m_2}{2}\ell_1^2\dot{\theta}_1^2 + \frac{m_2}{2}\ell_2^2\dot{\theta}_2^2 + m_2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)$$
$$+g(m_1 + m_2)\ell_1\cos\theta_1 + gm_2\ell_2\cos\theta_2$$

yields the equations of motion

$$\begin{aligned}0 &= m_2\ell_1\ell_2\dot\theta_1\dot\theta_2(-\sin\theta_1\cos\theta_2+\cos\theta_1\sin\theta_2)-g(m_1+m_2)\ell_1\sin\theta_1\\&\quad+(m_1+m_2)\ell_1^2\ddot\theta_1-m_2\ell_1\ell_2\frac{d}{dt}[\dot\theta_2(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2)],\\0 &= m_2\ell_1\ell_2\dot\theta_1\dot\theta_2(-\cos\theta_1\sin\theta_2+\sin\theta_1\cos\theta_2)-gm_2\ell_2\sin\theta_2\\&\quad-m_2\ell_2^2\ddot\theta_2-m_2\ell_1\ell_2\frac{d}{dt}[\dot\theta_1(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2)].\end{aligned}$$

**Example 4:** The description of a particle, moving in a spherically symmetric potential, $U(r)$, is the easiest to find in polar coordinate system,

$$\boldsymbol{x}=r\begin{pmatrix}\sin\theta\cos\phi\\\sin\theta\sin\phi\\\cos\theta\end{pmatrix},\quad \dot{\boldsymbol{x}}=\dot r\begin{pmatrix}\sin\theta\cos\phi\\\sin\theta\sin\phi\\\cos\theta\end{pmatrix}+r\begin{pmatrix}\dot\theta\cos\theta\cos\phi-\dot\phi\sin\theta\sin\phi\\\dot\theta\cos\theta\sin\phi+\dot\phi\sin\theta\cos\phi\\-\dot\theta\sin\theta\end{pmatrix}\tag{98}$$

where the Lagrangian is

$$L=\frac{m}{2}[\dot r^2+r^2\dot\theta^2+r^2\dot\phi^2\sin^2\theta]-U(r).\tag{99}$$

The equation of motion for $\theta$,

$$0=r^2\dot\phi^2\sin\theta\cos\theta-\frac{d}{dt}r^2\dot\theta\tag{100}$$

yields planar motion with $\theta=\frac{\pi}{2}$. The coordinate $\phi$ is cyclic and its generalized momentum,

$$p_\phi=mr^2\dot\phi\sin^2\theta,\tag{101}$$

the projection of the angular momentum on the $z$-axes, $p_\phi=L_z$, is conserved. The only non-trivial equation of motion is for the radius,

$$m\ddot r=-U'_{eff}(r),\tag{102}$$

where the effective potential,

$$U_{eff}(r)=U(r)+\frac{\ell^2}{2mr^2}\tag{103}$$

includes the centrifugal barrier. The radial equation of motion can be obtained from the Lagrangian of a one-dimensional system,

$$L=\frac{m}{2}\dot r^2-U_{eff}(r),\tag{104}$$

in particular the potential $U(r)=-\frac{e^2}{r}$ has stable orbits with non-vanishing angular momentum according to the effective potential of Fig. 34 (b).

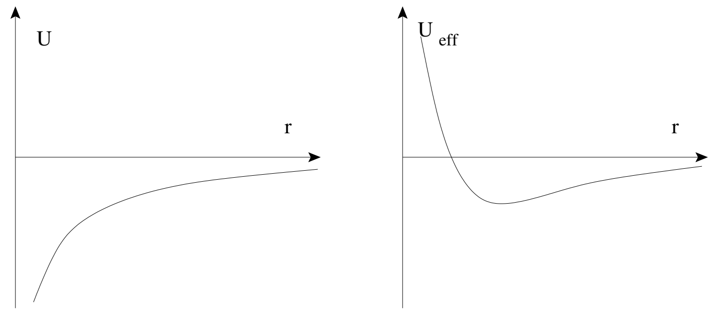



FIG. 34: (a): The Coulomb potential. (b): The effective potential with $\ell \neq 0$.

## C. Noether's theorem

The conservation of the generalized momentum of a cyclic coordinate suggests a relation between continuous symmetries and conservation laws because a continuous symmetry transformation leaves the Lagrangian unchanged hence its parameter, considered as a coordinate, is cyclic. In fact, Noether's theorem states that each continuous symmetry generates a conserved quantity. The symmetry, referred by Noether's theorem needs more precise definition. A symmetry transformation of the dynamics is supposed to preserve the form of the equation of motion. This latter is derived from the Lagrangian hence a symmetry is should preserve the Lagrangian. This is actually too strict condition, it can be relaxed to call the transformation $\boldsymbol{x}(t) \to \boldsymbol{x}'(t')$ a symmetry if the Lagrangian changes by a total time derivative, $L(\boldsymbol{x}, \dot{\boldsymbol{x}}) = L(\boldsymbol{x}', \dot{\boldsymbol{x}}') + \dot{\Lambda}(t', \boldsymbol{x}')$, the last term being absent in the variation equation. The symmetry transformation consist of a group and a transformation group is continuous if its elements can be parametrized by real numbers. One usually chooses the parametrization where the identity corresponds to vanishing parameters. The infinitesimal transformation, in the vicinity of the identity can therefore be written in the form $\boldsymbol{x} \to \boldsymbol{x} + \epsilon \boldsymbol{f}(t, \boldsymbol{x})$, $t \to t + \epsilon f(t, \boldsymbol{x})$ with infinitesimal $\epsilon$. The other new element in Noether's theorem, the conserved quantity, is a function of the coordinate and the velocity which remains constant on the physical trajectory, $\dot{F}(\boldsymbol{x}, \dot{\boldsymbol{x}}) = 0$. Noether's theorem is proven by using the infinitesimal symmetry transformation as a particular variation of the physical trajectory and by inspecting their Euler-Lagrange equation.

Let us consider only two kinds of symmetries for the sake of simplicity:

- $\boldsymbol{f} \neq 0$, $f = 0$: We consider the symmetry transformation $\boldsymbol{x} \to \boldsymbol{x} + \epsilon \boldsymbol{f}$ and the variation $\delta \boldsymbol{x} = \epsilon \boldsymbol{f}$ with a time dependent parameter, $\epsilon \to \epsilon(t)$. The Lagrangian for the transformed trajectory leads to the Lagrangian

$$\begin{aligned}\tilde{L}(\epsilon,\dot{\epsilon}) &= L(\boldsymbol{x}+\epsilon\boldsymbol{f},\dot{\boldsymbol{x}}+\epsilon\partial_t\boldsymbol{f}+\epsilon(\dot{\boldsymbol{x}}\boldsymbol{\partial})\boldsymbol{f}+\dot{\epsilon}\boldsymbol{f})+\mathcal{O}\left(\epsilon^2\right)\\ &= \epsilon\left[\frac{\partial L}{\partial\boldsymbol{x}}\boldsymbol{f}+\frac{\partial L}{\partial\dot{\boldsymbol{x}}}\partial_t\boldsymbol{f}+\frac{\partial L}{\partial\dot{\boldsymbol{x}}}(\dot{\boldsymbol{x}}\boldsymbol{\partial})\boldsymbol{f}\right]+\frac{\partial L}{\partial\dot{\boldsymbol{x}}}\dot{\epsilon}\boldsymbol{f}+\mathcal{O}\left(\epsilon^2\right).\end{aligned}$$

The variation principle, stated now as $\delta S[\epsilon]=\mathcal{O}\left(\epsilon^2\right)$, yields the Euler-Lagrange equation

$$\underbrace{\frac{\partial\tilde{L}(\epsilon,\dot{\epsilon})}{\partial\epsilon}}_{0}=\frac{d}{dt}\underbrace{\frac{\partial\tilde{L}(\epsilon,\dot{\epsilon})}{\partial\dot{\epsilon}}}_{p_\epsilon} \tag{105}$$

which states the conservation of the generalized momentum $p_\epsilon$ because the symmetry

$$L(\boldsymbol{x},\dot{\boldsymbol{x}})=L(\boldsymbol{x}+\epsilon\boldsymbol{f},\dot{\boldsymbol{x}}+\epsilon\partial_t\boldsymbol{f}+\epsilon(\dot{\boldsymbol{x}}\boldsymbol{\partial})\boldsymbol{f})+\mathcal{O}\left(\epsilon^2\right), \tag{106}$$

renders $\epsilon$ a cyclic coordinate.

**Examples:**

1. Translation symmetry, $\boldsymbol{f}=\boldsymbol{n}$, $\boldsymbol{n}^2=1$, of the Lagrangian $L=\frac{m}{2}\dot{\boldsymbol{x}}^2-U(\boldsymbol{T}\boldsymbol{x})$ with $T=\mathbb{1}-\boldsymbol{n}\otimes\boldsymbol{n}$, leads to the conservation of the momentum $p_\epsilon=m\dot{\boldsymbol{x}}\boldsymbol{n}$.
2. Rotational symmetry, $\boldsymbol{f}=\boldsymbol{n}\times\boldsymbol{x}$, $\boldsymbol{n}^2=1$ of the Lagrangian $L=\frac{m}{2}\dot{\boldsymbol{x}}^2-U(|\boldsymbol{x}|)$ implies the conservation of the angular momentum, $p_\epsilon=m\dot{\boldsymbol{x}}(\boldsymbol{n}\times\boldsymbol{x})=\boldsymbol{n}(\boldsymbol{x}\times m\dot{\boldsymbol{x}})=\boldsymbol{n}\boldsymbol{L}$.

- $\boldsymbol{f}=0$, $f\neq 0$: We use shifted time, $t\to t'=t+\epsilon$, and the trajectory $\boldsymbol{x}(t)\to\boldsymbol{x}(t-\epsilon)=\boldsymbol{x}(t)-\epsilon\dot{\boldsymbol{x}}(t)$ is subject of the variation $\delta\boldsymbol{x}=-\epsilon\dot{\boldsymbol{x}}$. The action, rewritten with a time-dependent $\epsilon$,

$$S[\boldsymbol{x}]=\int_{t_i+\epsilon(t_i)}^{t_f+\epsilon(t_f)}\frac{dt}{1+\dot{\epsilon}}L(\boldsymbol{x}(t-\epsilon),\dot{\boldsymbol{x}}(t-\epsilon)), \tag{107}$$

is $\epsilon$-independent by construction hence the $\mathcal{O}(\epsilon)$ variation is vanishing,

$$\begin{aligned}0 &= -\int_{t_i}^{t_f}dt\left(\epsilon\dot{\boldsymbol{x}}\frac{\partial L}{\partial\boldsymbol{x}}+\frac{d}{dt}(\epsilon\dot{\boldsymbol{x}})\frac{\partial L}{\partial\dot{\boldsymbol{x}}}+\dot{\epsilon}L\right)+\epsilon L(\boldsymbol{x}(t),\dot{\boldsymbol{x}}(t))\Big|_{t_i}^{t_f}\\ &= -\int_{t_i}^{t_f}dt\left[\epsilon\dot{\boldsymbol{x}}\left(\frac{\partial L}{\partial\boldsymbol{x}}-\frac{d}{dt}\frac{\partial L}{\partial\dot{\boldsymbol{x}}}\right)+\dot{\epsilon}L\right]+\epsilon\left(L-\dot{\boldsymbol{x}}\frac{\partial L}{\partial\dot{\boldsymbol{x}}}\right)_{t_i}^{t_f}.\end{aligned}$$

Since the trajectory solves the equation of motion we find for a time-independent $\epsilon(t)=\epsilon$ that

$$H=\frac{\partial L}{\partial\dot{\boldsymbol{x}}}\dot{\boldsymbol{x}}-L \tag{108}$$

is conserved.

### D. Relativistic particle

It is straightforward to extend the variation principle to a relativistic particle, the Lorentz invariant action can be function of the only invariant, the length of the world line. Dimensional reason and the appropriate non-relativistic limit leads to the choice

$$S = -mc\int_{s_i}^{s_f} ds. \tag{109}$$

This form of the action does not contain the world line, $x^\mu(s)$, thus one rewrites it in the form

$$S = -mc\int_{s_i}^{s_f} d\tau\sqrt{\frac{dx^\mu}{d\tau}g_{\mu\nu}\frac{dx^\mu}{d\tau}}. \tag{110}$$

Note that this integral is invariant under the reparametrization of the world line and the use of $\tau = t$ yields in the non-relativistic limit

$$S = -mc^2\int_{t_i}^{t_f} dt\sqrt{1-\frac{v^2}{c^2}} = -mc^2(t_f - t_i) + \int_{x_i}^{x_f} dt\frac{m}{2}v^2\left(1+\mathcal{O}\left(\frac{v^2}{c^2}\right)\right). \tag{111}$$

The momentum and the energy are

$$\begin{aligned}
\boldsymbol{p} &= \frac{\partial L}{\partial \boldsymbol{v}} = \frac{m\boldsymbol{v}}{\sqrt{1-\frac{v^2}{c^2}}},\\
E &= \boldsymbol{pv} - L = \frac{m\left[v^2 + c^2\left(1-\frac{v^2}{c^2}\right)\right]}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} = mc^2 + \frac{m}{2}v^2\left(1+\mathcal{O}\left(\frac{v^2}{c^2}\right)\right).
\end{aligned}$$

The parameter $\tau$ is not be the invariant length because it is advantageous to perform the variation $x^\mu(\tau) \to x^\mu(\tau) + \delta x^\mu(\tau)$ on each coordinate components independently. The variation leads to the Euler-Lagrange equation,

$$0 = -mc\frac{d}{d\tau}\frac{\frac{dx^\mu}{d\tau}}{\sqrt{\frac{dx^\rho}{d\tau}g_{\rho\nu}\frac{dx^\nu}{d\tau}}} = -mc\frac{\frac{d^2x^\mu}{d\tau^2}}{\sqrt{\frac{dx^\rho}{d\tau}g_{\rho\nu}\frac{dx^\nu}{d\tau}}} + mc\frac{\frac{dx^\mu}{d\tau}\frac{d^2x^\rho}{d\tau^2}g_{\rho\nu}\frac{dx^\nu}{d\tau}}{\sqrt{\left(\frac{dx^\rho}{d\tau}g_{\rho\nu}\frac{dx^\nu}{d\tau}\right)^3}} \tag{112}$$

where we are free to use the proper length as parameter, $\tau \to s$, leading to a significantly simpler equation,

$$0 = \ddot{x}^\mu. \tag{113}$$

The four-momentum is

$$p_\mu = -\frac{\partial L}{\partial \dot{x}^\mu} = mc\dot{x}_\mu, \quad p^\mu = mc\dot{x}^\mu. \tag{114}$$

## VII. FIELD THEORIES

### A. A mechanical model

A simple mechanical model of a scalar field in 1+1 dimensional space-time can be given by considering a chain of pendules, coupled to their neighbours by a spring, depicted in Fig. 35. The Lagrangian of the system of pendules is given by

$$L = \sum_n \left[ \frac{mr^2}{2} \dot{\theta}_n^2 - \frac{kr_0^2}{2} (\theta_{n+1} - \theta_n)^2 - gr \cos\theta_n \right]. \tag{115}$$

The variable transformation $\theta_n(t) \to \Phi\theta_n(t) = \phi(t, x_n)$ brings the Lagrangian into the form

$$L = a \sum_n \left[ \frac{1}{2c^2} (\partial_t \phi_n)^2 - \frac{1}{2} \left( \frac{\phi_{n+1} - \phi_n}{a} \right)^2 - \lambda \cos\frac{\phi_n}{\Phi} \right] \tag{116}$$

with $\Phi = r_0\sqrt{ak}$, $c = a\frac{r_0}{r}\sqrt{\frac{k}{m}}$, $\lambda = \frac{gr}{a}$. We perform the continuum limit, $a \to 0$,

$$L = \int dx \left[ \frac{1}{2c^2} (\partial_t \phi(x))^2 - \frac{1}{2} (\partial_x \phi(x))^2 - \lambda \cos\frac{\phi(x)}{\Phi} \right] \tag{117}$$

and find the action of the so called sine-Gordon model,

$$S = \int dt dx \left[ \frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) - \lambda \cos\frac{\phi(x)}{\Phi} \right]. \tag{118}$$

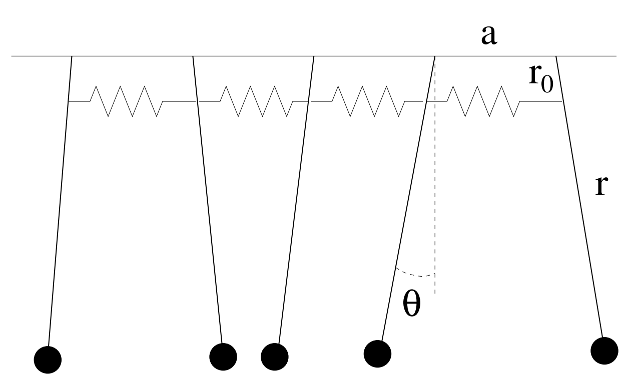



FIG. 35: The mechanical model of a 1+1 dimensional scalar field.

### B. Equation of motion

Let us consider a scalar field theory whose equation of motion can be uniquely solved by imposing the auxiliary conditions, $\phi(t_i, \boldsymbol{x}) = \phi_i(\boldsymbol{x})$, $\phi(t_f, \boldsymbol{x}) = \phi_f(\boldsymbol{x})$. The action funcitonal which is local in space-time can be written as an integral,

$$S[\phi(\cdot)] = \int_V \underbrace{dtd^3x}_{\frac{1}{c}d^4x} L(\phi, \partial\phi). \tag{119}$$

We perform the variation $\phi(x) \to \phi(x) + \delta\phi(x)$, $\delta\phi(t_i, \boldsymbol{x}) = \delta\phi(t_f, \boldsymbol{x}) = 0$, the corresponding variation of the action is

$$\begin{aligned}
\delta S &= \int_V dx \left( \frac{\partial L(\phi, \partial\phi)}{\partial\phi_a} \delta\phi_a + \frac{\partial L(\phi, \partial\phi)}{\partial\partial_\mu\phi_a} \delta\partial_\mu\phi_a \right) + \mathcal{O}\left(\delta^2\phi\right) \\
&= \int_V dx \left( \frac{\partial L(\phi, \partial\phi)}{\partial\phi_a} \delta\phi_a + \frac{\partial L(\phi, \partial\phi)}{\partial\partial_\mu\phi_a} \partial_\mu\delta\phi_a \right) + \mathcal{O}\left(\delta^2\phi\right) \\
&= \int_{\partial V} ds^\mu \delta\phi_a \frac{\partial L(\phi, \partial\phi)}{\partial\partial_\mu\phi_a} + \int_V dx \delta\phi_a \left( \frac{\partial L(\phi, \partial\phi)}{\partial\phi_a} - \partial_\mu \frac{\partial L(\phi, \partial\phi)}{\partial\partial_\mu\phi_a} \right) + \mathcal{O}\left(\delta^2\phi\right).
\end{aligned}$$

The surface contribution is vansihing for $\mu = 0$,

$$\int_{\partial V} ds^0 \delta\phi_a \frac{\partial L(\phi, \partial\phi)}{\partial\partial_0\phi_a} = \int_{t=t_f} d^3x \underbrace{\delta\phi_a}_{0} \frac{\partial L(\phi, \partial\phi)}{\partial\partial_0\phi_a} - \int_{t=t_i} d^3x \underbrace{\delta\phi_a}_{0} \frac{\partial L(\phi, \partial\phi)}{\partial\partial_0\phi_a} = 0 \tag{120}$$

due to the absence of variation at the initial and the final time can be omitted for $\mu = j$,

$$\int_{\partial V} ds^j \delta\phi_a \frac{\partial L(\phi, \partial\phi)}{\partial\partial_j\phi_a} = \int_{x_j=\infty} ds^j \delta\phi_a \frac{\partial L(\phi, \partial\phi)}{\partial\partial_j\phi_a} - \int_{x_j=-\infty} ds^j \delta\phi_a \frac{\partial L(\phi, \partial\phi)}{\partial\partial_j\phi_a} \tag{121}$$

in deriving the equation of motion at a fixed, finite space location. The resulting Euler-Lagrange equation is

$$\frac{\partial L(\phi, \partial\phi)}{\partial\phi_a} - \partial_\mu \frac{\partial L(\phi, \partial\phi)}{\partial\partial_\mu\phi_a} = 0. \tag{122}$$

### C. Wave equations for a scalar particle

The wave equations represent an important family of equations of motion in field theory. The Schrödinger equation

$$i\hbar\partial_t\psi(t, \boldsymbol{x}) = \left[ -\frac{\hbar^2}{2m}\Delta + U(\boldsymbol{x}) \right] \psi(t, \boldsymbol{x}), \tag{123}$$

can be derived from the Lagrangian

$$L = \psi^* \left[ i\hbar\partial_t + \frac{\hbar^2}{2m}\Delta - U(\boldsymbol{x}) \right] \psi. \tag{124}$$

Its relativistic generalisation, the second order Kleing Gordon equation, can be found by using the four momentum operator $\hat{p}_\mu = (\frac{E}{c}, -\boldsymbol{p}) = -\frac{\hbar}{i}\partial_\mu$ or $\hat{p}^\mu = (\frac{E}{c}, \boldsymbol{p}) = i\hbar\partial^\mu$, in writing the mass-shell condition, $p^2 = m^2c^2$, as a differential equation,

$$0 = (\hat{p}^2 - m^2c^2)\phi = -\hbar^2 \left( \partial_\mu\partial^\mu + \frac{m^2c^2}{\hbar^2} \right) \phi. \tag{125}$$

This equation can be simplified by introducing the Compton wavelength of the particle, $\lambda_C = \frac{\hbar}{mc}$, as

$$0 = \left(\Box + \frac{1}{\lambda_C^2}\right)\phi. \tag{126}$$

The Lagrangian, leading to the gemneralization of this equation for the self-interacting particles is

$$L = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2\lambda_C^2}\phi^2 - U(\phi) \tag{127}$$

for real $\phi(x)$ and

$$L = \partial\phi^*\partial\phi - \frac{1}{\lambda_C^2}\phi^*\phi - V(\phi^*\phi), \tag{128}$$

in the complex case.

### D. Electrodynamics

Charge: $x_a^\mu(s)$, $a = 1, \ldots, n$, EM field: $A^\mu(x)$

**Charge:**

$$\begin{aligned} S_{ch} &= -\int_{x_i}^{x_f}\left(mcds + \frac{e}{c}A_\mu dx^\mu\right) = \int_{\tau_i}^{\tau_f} L_\tau d\tau \\ L_\tau &= -mc\sqrt{x'^2} - \frac{e}{c}A_\mu(x)x'^\mu \end{aligned}$$

Gauge transformation: $A_\mu(x) \to A_\mu(x) + \partial_\mu\Phi(x)$

Gauge invariance: $\Delta S = \frac{e}{c}[\Phi(x_i) - \Phi(x_f)]$ drops out from the equations of motion

Euler-Lagrange equation:

$$\begin{aligned} 0 &= \frac{\partial L_\tau}{\partial x^\mu} - \frac{d}{d\tau}\frac{\partial L_\tau}{\partial x'^\mu} = -\frac{e}{c}\partial_\mu A_\nu(x)x'^\nu + mc\frac{d}{d\tau}\frac{x'_\mu}{\sqrt{x'^2}} + \frac{e}{c}\frac{d}{d\tau}A_\mu(x) \\ \tau \to s: \quad mc\ddot{x}_\nu &= \frac{e}{c}F_{\mu\nu}\dot{x}^\nu \end{aligned}$$

Field strength tensor:

$$F_{\mu\nu} = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x). \tag{129}$$

Electric current:

$$\begin{aligned} j^\mu(x) &= c\sum_a\int ds\delta(x - x_a(s))\dot{x}^\mu \\ &= c\sum_a\int ds\delta(\boldsymbol{x} - \boldsymbol{x}_a(s))\delta(x^0 - x_a^0(s))\dot{x}^\mu \\ &= c\sum_a\delta(\boldsymbol{x} - \boldsymbol{x}_a(s))\frac{\dot{x}^\mu}{|\dot{x}^0|} \end{aligned}$$

$$= \underbrace{\sum_a \delta(\boldsymbol{x} - \boldsymbol{x}_a(s))}_{\rho(\boldsymbol{x})} \frac{dx^\mu}{dt}.$$

Current conservation:

$$\partial_\mu j^\mu = c\partial_0 \rho + \nabla \cdot \boldsymbol{j} = \sum_a e_a[-\boldsymbol{v}_a(t)\boldsymbol{\nabla}\delta(\boldsymbol{x} - \boldsymbol{x}_a(t)) + \boldsymbol{\nabla}\delta(\boldsymbol{x} - \boldsymbol{x}_a(t))\boldsymbol{v}_a(t)] = 0 \tag{130}$$

**EM field:**

Lagrangian:

1. $L_A = \mathcal{O}\left((\partial_0 A_\mu)^2\right)$
2. invariance de Lorentz
3. invariance de jauge

$$L_A = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} \tag{131}$$

Full action:

$$\begin{aligned} S_{EM} &= -\sum_a \left[ mc \int ds \sqrt{\dot{x}_a^\mu g_{\mu\nu} \dot{x}_a^\nu} + \frac{e}{c} \int A_\mu(x) dx^\mu \right] - \frac{1}{16\pi c} \int F^{\mu\nu} F_{\mu\nu} dx \\ &= -mc \sum_a \int ds \sqrt{\dot{x}_a^\mu g_{\mu\nu} \dot{x}_a^\nu} - \frac{1}{c} \int \left[ \frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} + \frac{e}{c} j^\mu A_\mu \right] dx \end{aligned}$$

Maxwell: $(\boldsymbol{E}, \boldsymbol{H}) \to A_\mu$, the first unification in physics

$$\begin{aligned} 0 &= \frac{\delta L}{\delta A_\mu} - \partial_\nu \frac{\delta L}{\delta \partial_\nu A_\mu}, \qquad F^{\mu\nu} F_{\mu\nu} = 2\partial_\mu A_\nu \partial^\mu A^\nu - 2\partial_\mu A_\nu \partial^\nu A^\mu \\ &= -\frac{e}{c} j^\mu + \frac{1}{4\pi} \partial_\nu (\partial^\nu A^\mu - \partial^\mu A^\nu) \\ \frac{1}{4\pi} \partial_\nu F^{\nu\mu} &= \frac{e}{c} j^\mu \end{aligned}$$

Bianchi identity : $\partial_\mu \partial_\nu A_\rho = \partial_\nu \partial_\mu A_\rho$

$$\partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} + \partial_\mu F_{\nu\rho} = 0 \tag{132}$$

Nonrelativistic parametrisation: $A^\mu = (\phi, \boldsymbol{A})$, $A_\mu = (\phi, -\boldsymbol{A})$

$$\begin{aligned} \boldsymbol{E} &= -\partial_0 \boldsymbol{A} - \nabla\phi = -\frac{1}{c}\partial_t \boldsymbol{A} - \nabla\phi, \\ \boldsymbol{H} &= \nabla \times \boldsymbol{A}. \end{aligned}$$

Inversion:

$$\epsilon_{jk\ell} H_\ell = \epsilon_{jk\ell} \epsilon_{\ell mn} \nabla_m A_n = (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) \nabla_m A_n = \nabla_j A_k - \nabla_k A_j = -F_{jk} \tag{133}$$

Field strength:

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -H_z & H_y \\ -E_y & H_z & 0 & -H_x \\ -E_z & -H_y & H_x & 0 \end{pmatrix}, \quad F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -H_z & H_y \\ E_y & H_z & 0 & -H_x \\ E_z & -H_y & H_x & 0 \end{pmatrix} \tag{134}$$

Dual field strength:

$$\begin{aligned}
\tilde{F}_{\mu\nu} &= \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} \\
\tilde{F}_{0j} &= -\frac{1}{2}\epsilon_{jk\ell}F^{k\ell} = \frac{1}{2}\epsilon_{jk\ell}\epsilon_{k\ell m}H_m = H_j \quad (\epsilon_{123} = -\epsilon^{123} = -1) \\
\tilde{F}_{jk} &= -\epsilon_{jk\ell}F^{\ell 0} = \epsilon_{jk\ell}E_\ell \\
\tilde{F}_{\mu\nu} &= \begin{pmatrix} 0 & H_x & H_y & H_z \\ -H_x & 0 & E_z & -E_y \\ -H_y & -E_z & 0 & E_x \\ -H_z & E_y & -E_x & 0 \end{pmatrix}, \quad \tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -H_x & -H_y & -H_z \\ H_x & 0 & E_z & -E_y \\ H_y & -E_z & 0 & E_x \\ H_z & E_y & -E_x & 0 \end{pmatrix}
\end{aligned}$$

Inhomogeneous Maxwell equations:

$$\begin{aligned}
\frac{4\pi}{c}j^0 = \nabla_j F^{j0}, \quad &\rightarrow \quad 4\pi\rho = \boldsymbol{\nabla}\boldsymbol{E} \\
\frac{4\pi}{c}j^k = \partial_0 F^{0k} + \nabla_j F^{jk} \quad &\rightarrow \quad \frac{4\pi}{c}\boldsymbol{j} = -\frac{1}{c}\partial_t\boldsymbol{E} + \boldsymbol{\nabla}\times\boldsymbol{H}
\end{aligned}$$

Homogeneous Maxwell equations:

$$\begin{aligned}
0 &= \partial_\mu \tilde{F}^{\mu\nu} \\
0 &= \nabla_j \tilde{F}^{j0} = \boldsymbol{\nabla}\boldsymbol{H}, \\
0 &= \partial_0 \tilde{F}^{0k} + \nabla_j \tilde{F}^{jk} = -\left(\frac{1}{c}\partial_t\boldsymbol{H} + \boldsymbol{\nabla}\times\boldsymbol{E}\right)
\end{aligned}$$