Quantum Mechanics II.

Janos Polonyi

University of Strasbourg

(Dated: October 26, 2020)

Video: `https://drive.google.com/drive/folders/11u3UNyrS6b-jXhRChaWBC3SXaZOz9yu_?usp=sharing`

Contents

Quantum mechanics is usually taught on four different levels:

1. Basic ideas, simple examples for a one dimensional particle, particle in spherical potential

2. More realistic, three dimensional cases with few particles $\leftarrow$

3. Several particles, relativistic effects (Quantum Field Theory)

4. Fundamental issues, challenges, paradoxes and interpretation of the quantum world

I. PERTURBATION EXPANSION

$$H = H_0 + gH_1$$

A. Stationary perturbations

- **Example:** Atom in an EM field

- **Goal:** solve stationary Schrödinger equation, $H|\psi_n\rangle = E_n|\psi_n\rangle$

- **Expansion in g:**

$$\begin{aligned}
|\psi_n\rangle &= |\psi_n^{(0)}\rangle + g|\psi_n^{(1)}\rangle + g^2|\psi_n^{(2)}\rangle + \cdots , \\
E_n &= E_n^{(0)} + gE_n^{(1)} + g^2E_n^{(2)} + \cdots , \\
0 &= (H_0 + gH_1 - E_n^{(0)} - gE_n^{(1)} - g^2E_n^{(2)} - \cdots)(|\psi_n^{(0)}\rangle + g|\psi_n^{(1)}\rangle + g^2|\psi_n^{(2)}\rangle + \cdots) \\
&= g^0\left(H_0|\psi_n^{(0)}\rangle - E_n^{(0)}|\psi_n^{(0)}\rangle\right) \\
&\quad +g\left(H_0|\psi_n^{(1)}\rangle + H_1|\psi_n^{(0)}\rangle - E_n^{(1)}|\psi_n^{(0)}\rangle - E_n^{(0)}|\psi_n^{(1)}\rangle\right) \\
&\quad +g^2\left(H_0|\psi_n^{(2)}\rangle + H_1|\psi_n^{(1)}\rangle - E_n^{(2)}|\psi_n^{(0)}\rangle - E_n^{(1)}|\psi_n^{(1)}\rangle - E_n^{(0)}|\psi_n^{(2)}\rangle\right) + \cdots
\end{aligned}$$

  Orders one-by-one:

$$\begin{aligned}
\mathcal{O}\left(g^0\right): &\quad H_0|\psi_n^{(0)}\rangle &&= E_n^{(0)}|\psi_n^{(0)}\rangle \\
\mathcal{O}\left(g\right): &\quad (H_0 - E_n^{(0)})|\psi_n^{(1)}\rangle &&= (E_n^{(1)} - H_1)|\psi_n^{(0)}\rangle \\
\mathcal{O}\left(g^2\right): &\quad (H_0 - E_n^{(0)})|\psi_n^{(2)}\rangle &&= (E_n^{(1)} - H_1)|\psi_n^{(1)}\rangle + E_n^{(2)}|\psi_n^{(0)}\rangle \\
\mathcal{O}\left(g^k\right): &\quad (H_0 - E_n^{(0)})|\psi_n^{(k)}\rangle &&= (E_n^{(1)} - H_1)|\psi_n^{(k-1)}\rangle + E_n^{(2)}|\psi_n^{(k-2)}\rangle + \cdots + E_n^{(k)}|\psi_n^{(0)}\rangle
\end{aligned}$$

- **Zeroth order:** unperturbed stationary states, $|\psi_n^{(0)}\rangle$, $\langle\psi_m^{(0)}|\psi_n^{(0)}\rangle = \delta_{mn}$

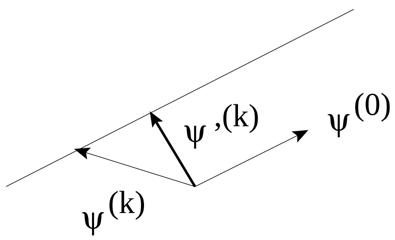
- **Higher order:** no unique solution

  1. $(H_0 - E_n^{(0)})^{-1}$ does not esists in the null space of $H_0 - E_n^{(0)}$

     $\nwarrow$

     $$\mathcal{N}_A = \{|\psi\rangle | A|\psi\rangle = 0\}$$

  2. Another way to see: if $|\psi_n^{(k)}\rangle$ ($k > 0$ is a solution $\Longrightarrow |\psi_n'^{(k)}\rangle = |\psi_n^{(k)}\rangle + c|\psi_n^{(0)}\rangle$ is another solution
  3. Unique solution: choose $c = -\langle\psi_n^{(0)}|\psi_n^{(k)}\rangle \Longrightarrow \langle\psi_n^{(0)}|\psi_n'^{(k)}\rangle = \langle\psi_n^{(0)}|(|\psi_n^{(k)}\rangle - |\psi_n^{(0)}\rangle\langle\psi_n^{(0)}|\psi_n^{(k)}\rangle) = 0$



- **First order:** One writes $|\psi_n^{(1)}\rangle = \sum_\ell c_{n,\ell}|\psi_\ell^{(0)}\rangle$

$$\begin{aligned}
\langle\psi_k^{(0)}| \qquad (H_0 - E_n^{(0)})|\psi_n^{(1)}\rangle &= (E_n^{(1)} - H_1)|\psi_n^{(0)}\rangle \\
\sum_\ell c_{n,\ell}\langle\psi_k^{(0)}|(H_0 - E_n^{(0)})|\psi_\ell^{(0)}\rangle &= \langle\psi_k^{(0)}|(E_n^{(1)} - H_1)|\psi_n^{(0)}\rangle \\
\sum_\ell c_{n,\ell}\langle\psi_k^{(0)}|(E_k^{(0)} - E_n^{(0)})|\psi_\ell^{(0)}\rangle &= \langle\psi_k^{(0)}|(E_n^{(1)} - H_1)|\psi_n^{(0)}\rangle \\
\langle\psi_k^{(0)}|\psi_\ell^{(0)}\rangle = \delta_{k,\ell} \quad \rightarrow \quad (E_k^{(0)} - E_n^{(0)})c_{n,k} &= E_n^{(1)}\delta_{k,n} - H_{1kn} \quad \leftarrow \quad \langle\psi_m^{(0)}|H_1|\psi_n^{(0)}\rangle
\end{aligned}$$

Solution:

$$\boxed{c_{n,k} = \begin{cases} \frac{H_{1kn}}{E_n^{(0)} - E_k^{(0)}}, & k \neq n \\ 0 & k = n, \end{cases}}$$

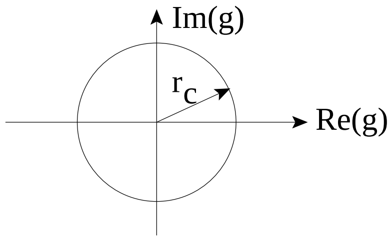
$$\boxed{E_n^{(1)} = H_{1nn}}$$

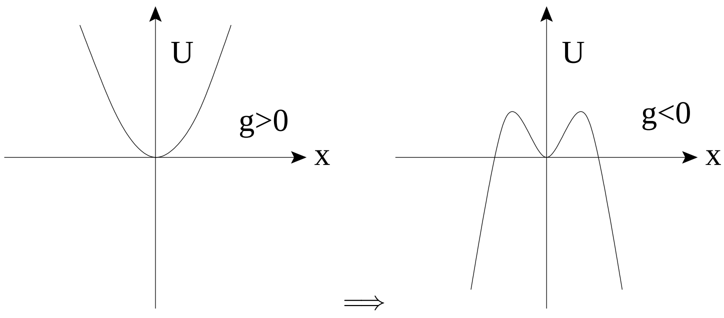
- **Necessary conditions:**

$$\begin{aligned}
g\langle\psi_n^{(0)}|H_1|\psi_n^{(0)}\rangle &\ll E_n^{(0)} \\
g|\langle\psi_k^{(0)}|H_1|\psi_n^{(0)}\rangle| &\ll |E_n^{(0)} - E_k^{(0)}|.
\end{aligned}$$

- **Convergence radius:** $r_c = 0$

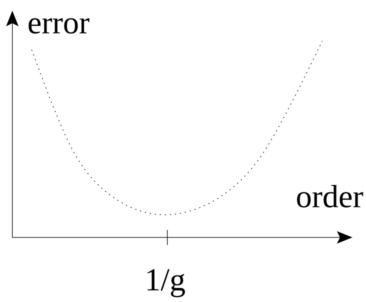
$$H = \frac{p^2}{2m} + U(x), \qquad U(x) = \frac{m\omega^2}{2}x^2 + \frac{g}{4!}x^4, \quad g \to -g\ ?$$





- **Asymptotic convergence:**

  1. Defintion: $f_N(g) = \sum_{n=0}^{N} f_n g^n \rightarrow_{as} f(g)$ if $\frac{f_N(g)-f(g)}{g^N} \rightarrow 0$ as $g \rightarrow 0$
  2. Quantum mechanical systems: $|f_N(g) - f(g)|$ starts to grow at $N = \mathcal{O}\left(\frac{1}{g}\right)$ (QED: $g = \frac{1}{137}$)



- **Degenerate perturbations:**

  1. *Problem:*
     (a) $H_0|\psi_n^{(0)}\rangle = E_n^{(0)}|\psi_n^{(0)}\rangle \Longrightarrow |\psi_n^{(0)}\rangle$ is ill defined within the degenerate subspace
     (b) *The higher orders* in $|\psi_n\rangle = |\psi_n^{(0)}\rangle + g|\psi_n^{(1)}\rangle + g^2|\psi_n^{(2)}\rangle + \cdots$ are not small
     (c) *Singularity* at $g = 0$
     (d) $g|\langle\psi_k^{(0)}|H_1|\psi_n^{(0)}\rangle| \ll |E_n^{(0)} - E_k^{(0)}|$ is violated
  2. *Solution:* diagonalize H_1 within the degenerate subspace

3. *Degeneracy:* $E_k^{(0)} = E_\ell^{(0)}$ for $1 \le k, \ell \le N \ll \dim(H)$

$$H_1 = \begin{pmatrix} \begin{pmatrix} H_{1\ 1,1} & 0 & \cdots & 0 \\ 0 & H_{1\ 2,2} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & H_{1\ N,N} \end{pmatrix} & B \\ B^\dagger & H_1' \end{pmatrix},$$

and suppose that $H_{1\ j,j} \ne H_{1\ k,k}$ for $j \ne k$.

4. *Secular equation:*

   - Eigenvalues: $A|\psi\rangle = a|\psi\rangle \quad \leftrightarrow \quad (A - a\mathbb{1})|\psi\rangle = 0 \quad \leftrightarrow \quad \det(A - a\mathbb{1}) = 0$
   - $\det[H_{1\ k\ell} - \delta_{k,\ell} E_k^{(1)}] = 0 \leftrightarrow E_k^{(1)} = H_{1\ kk}$

5. *Higher orders are regular:*

$$\begin{aligned} |\psi_k\rangle &= |\psi_k^{(0)}\rangle + \mathcal{O}(g) \\ E_k &= E_k^{(0)} + gH_{1kk} + \mathcal{O}\left(g^2\right) \end{aligned}$$

6. *Physical importance:*

   (a) The increased sensitivity of the eigenfunctions on the perturbations: large $|\frac{\langle\psi_n^{(0)}|H_1|\psi_k^{(0)}\rangle}{E_n^{(0)} - E_k^{(0)}}|$
   Weak interactions become more important for exact or approximate degeneracy

   (b) An atom interacting with an ideal gas in box L:

   - Typical level spacing of the gas: $\Delta E \sim \frac{\hbar^2}{mL^2}$
   - "Small" parameter of the perturbation expansion:

$$\frac{gH_{1kn}}{\frac{\hbar^2}{mL^2}} \sim 10^{54} mL^2 gH_{1kn} > 1$$

   (m, L expressed gram and centimeter)

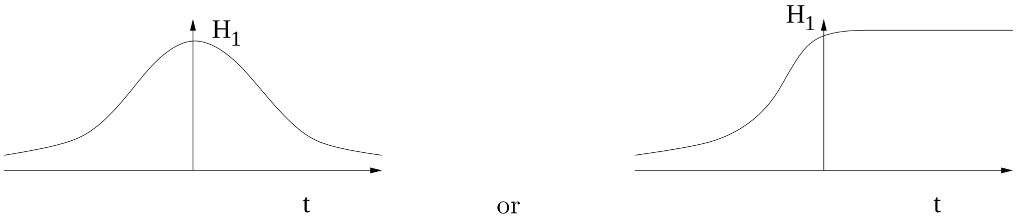
   - Classical limit in quantum mechanics
   - Relaxation in statistical physics

B. Time dependent perturbations

1. **Goal:**

$$H = H_0 + gH_1(t),$$

With initial condition: $|\psi(t = -\infty)\rangle = |\psi_n^{(0)}\rangle$



Transition probability:

$$\boxed{P_{n\to k}(t) = |\langle\psi_k^{(0)}|\psi(t)\rangle|^2}$$

2. **Solution:**

$$|\psi(t)\rangle = \sum_k c_k(t)|\psi_k^{(0)}(t)\rangle$$

In time dependent basis:

$$\begin{aligned}
i\hbar\partial_t|\psi_k^{(0)}(t)\rangle &= H_0|\psi_k^{(0)}(t)\rangle \\
H_0|\psi_k^{(0)}(0)\rangle &= E_k^{(0)}|\psi_k^{(0)}(0)\rangle \\
|\psi_k^{(0)}(t)\rangle &= e^{-i\frac{t}{\hbar}E_k^{(0)}}|\psi_k^{(0)}(0)\rangle
\end{aligned}$$

3. **Schrödinger equation:**

$$\begin{aligned}
i\hbar\partial_t|\psi(t)\rangle &= [H_0 + gH_1(t)]|\psi(t)\rangle \\
i\hbar\sum_k(\partial_t c_k(t)|\psi_k^{(0)}(t)\rangle + c_k(t)\underbrace{\partial_t|\psi_k^{(0)}(t)\rangle}_{\frac{1}{i\hbar}H_0|\psi_k^{(0)}(t)\rangle}) &= [H_0 + gH_1(t)]|\sum_k c_k(t)|\psi_k^{(0)}(t)\rangle \\
\langle\psi_\ell^{(0)}| \qquad i\hbar\partial_t c_\ell(t) &= g\sum_k\langle\psi_\ell^{(0)}(t)|H_1(t)|\psi_k^{(0)}(t)\rangle c_k(t) = g\sum_k c_k(t)H_{1\ell k}(t) \\
c_\ell(t) &= \sum_k g^k c_\ell^{(k)}(t)
\end{aligned}$$

Order by order:

$$\begin{aligned}
\mathcal{O}\left(g^0\right): \ i\hbar\partial_t c_\ell^{(0)}(t) = &\ 0, \\
\mathcal{O}\left(g^m\right): \ i\hbar\partial_t c_\ell^{(m)}(t) = &\sum_k H_{1\ell k}(t)c_k^{(m-1)}(t).
\end{aligned}$$

4. **Solution:** $H_1(t) = f(t)H'$

$$\begin{aligned}
c_k^{(0)} &= \delta_{k,n} \\
c_k(t) &= \delta_{k,n} - \frac{ig}{\hbar}\int_{-\infty}^t dt' H_{1kn}(t') + \mathcal{O}\left(g^2\right) \\
H_{1\ell k}(t) &= e^{i\frac{t}{\hbar}E_\ell^{(0)}}\langle\psi_\ell^{(0)}(0)|H'|\psi_k^{(0)}(0)\rangle e^{-i\frac{t}{\hbar}E_k^{(0)}}f(t) = H'_{\ell k}e^{i\omega_{\ell k}t}f(t) \\
\hbar\omega_{\ell k} &= E_\ell^{(0)} - E_k^{(0)}, \qquad H'_{\ell k} = \langle\psi_\ell^{(0)}(0)|H'|\psi_k^{(0)}(0)\rangle \\
c_k(t) &= \delta_{k,n} - \frac{igH'_{kn}}{\hbar}\int_{-\infty}^t dt' f(t')e^{i\omega_{k,n}t'} + \mathcal{O}\left(g^2\right)
\end{aligned}$$

5. Transition probability:

$$\boxed{P_{n(\neq k)\to k}(t) = |c_k(t)|^2 = \left|\frac{gH'_{kn}}{\hbar}\right|^2 \left|\int_{-\infty}^{t} dt' f(t') e^{i\omega_{kn}t'}\right|^2 + \mathcal{O}\left(g^3\right)}$$

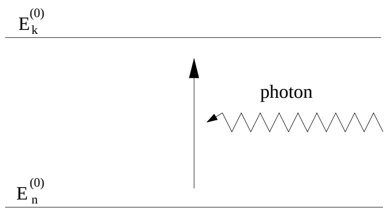
6. **Example:** Sinusoidal perturbation is turned on suddenly

- Transition amplitude:

$$\begin{aligned}
f(t) &= \begin{cases} 2\cos\omega t, & \omega > 0 \quad t > 0 \\ 0 & \qquad\quad\; t < 0, \end{cases} \\
c_{k\neq n} &= -\frac{igH'_{kn}}{\hbar}\int_0^t dt' e^{i\omega_{k,n}t'}\left(e^{i\omega t'} + e^{-i\omega t'}\right) \\
&= -\frac{gH'_{kn}}{\hbar}\left(\frac{e^{i(\omega_{k,n}-\omega)t}-1}{\omega_{k,n}-\omega} + \frac{e^{i(\omega_{k,n}+\omega)t}-1}{\omega_{k,n}+\omega}\right) \qquad \left[\int dt e^{i\omega t} = \frac{e^{i\omega t}}{i\omega}\right] \\
e^{i\phi}-1 &= e^{i\frac{\phi}{2}}\left(e^{i\frac{\phi}{2}} - e^{-i\frac{\phi}{2}}\right) = 2ie^{i\frac{\phi}{2}}\sin\frac{\phi}{2} \\
c_{k\neq n} &= -\frac{2igH'_{k,n}}{\hbar}\left(\frac{e^{\frac{i}{2}(\omega_{kn}-\omega)t}\sin\frac{\omega_{kn}-\omega}{2}t}{\omega_{kn}-\omega} + \frac{e^{\frac{i}{2}(\omega_{kn}+\omega)t}\sin\frac{\omega_{kn}+\omega}{2}t}{\omega_{kn}+\omega}\right)
\end{aligned}$$

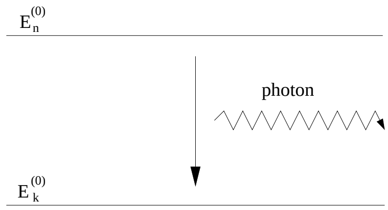
- Transition probability for $\omega \approx \omega_{kn}$:

$$\begin{aligned}
P &\approx \begin{cases} P^- & \omega_{kn} > 0 \text{ (absorption)}, \\ P^+ & \omega_{kn} < 0 \text{ (emission)}, \end{cases} \\
P^{\pm}_{n\to k} &= \frac{4g^2|H'_{k,n}|^2}{\hbar^2(\omega_{kn}\pm\omega)^2}\sin^2\frac{1}{2}(\omega_{kn}\pm\omega)t.
\end{aligned}$$



(a)

absorption: $P^-_{n\to k} = P(n+\gamma\to k)$



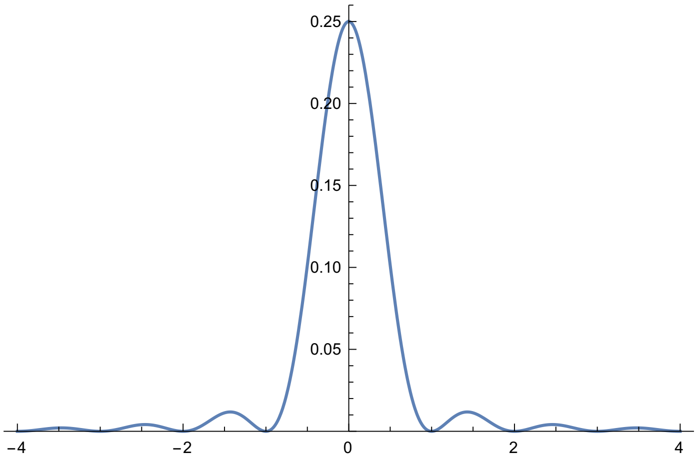
(b)

emission: $P^+_{n\to k} = P(n\to k+\gamma)$

- Small and large t asymptotics:

$$\begin{aligned}
t\approx 0:\ P^{\pm}_{n\to k} &= t^2\frac{|gH_{kn}|^2}{\hbar^2} \\
t\to\infty:\ w^{\pm}_{n\to k} &= \frac{P^{\pm}_{n\to k}}{t} = \frac{2\pi|gH_{kn}|^2}{\hbar^2}\frac{2}{\pi}\frac{\sin^2\frac{t(\omega_{kn}\pm\omega)}{2}}{t(\omega_{kn}\pm\omega)^2} = \frac{2\pi|gH_{kn}|^2}{\hbar^2}\delta_t(\omega_{kn}\pm\omega) \\
\delta(x) &= \frac{2}{\pi}\lim_{\eta\to\infty}\frac{\sin^2\frac{\eta x}{2}}{\eta x^2}
\end{aligned}$$

$\frac{\sin^2\pi x}{(2\pi x)^2}$:



C. Non-exponential decay rate

1. **Time dependence:**

- Initial state: $|\psi_{in}\rangle$ at $t = 0$, $H|\psi_{in}\rangle \neq E|\psi_{in}\rangle$
- Time evolution:

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}Ht}|\psi_{in}\rangle$$

- Probability to preserve the initial state:

$$P_0(t) = |A(t)|^2$$

- Persistence amplitude:

$$A(t) = \langle\psi_{in}|e^{-\frac{i}{\hbar}Ht}|\psi_{in}\rangle.$$

- The decay is usually not exponential and has short, intermediate and long time regimes.

2. **Short time regime:**

- Persistence amplitude:

$$\begin{aligned} A(t) &= 1 - \frac{it}{\hbar}\langle\psi_{in}|H|\psi_{in}\rangle - \frac{t^2}{2\hbar^2}\langle\psi_{in}|H^2|\psi_{in}\rangle + \mathcal{O}\left(t^3\right) \\ P_0(t) &= \left(1 - \frac{it}{\hbar}\langle\psi_{in}|H|\psi_{in}\rangle - \frac{t^2}{2\hbar^2}\langle\psi_{in}|H^2|\psi_{in}\rangle\right)\left(1 + \frac{it}{\hbar}\langle\psi_{in}|H|\psi_{in}\rangle - \frac{t^2}{2\hbar^2}\langle\psi_{in}|H^2|\psi_{in}\rangle\right) + \mathcal{O}\left(t^3\right) \\ &= 1 + \frac{t^2}{\hbar^2}\langle\psi_{in}|H|\psi_{in}\rangle^2 - \frac{t^2}{\hbar^2}\langle\psi_{in}|H^2|\psi_{in}\rangle + \mathcal{O}\left(t^3\right) \\ &= 1 - \frac{t^2}{t_Z^2} + \mathcal{O}\left(t^3\right), \qquad t_Z = \frac{\hbar}{\sqrt{\langle\psi_{in}|(H - \langle\psi_{in}|H|\psi_{in}\rangle)^2|\psi_{in}\rangle}} \quad \leftarrow \quad \text{Zeno time} \end{aligned}$$

3. **Intermediate time regime:**

- Projection operators: $P^\dagger = P$, $P^2 = P$ (spectrum$= \{0,1\}$)

  (a) Longitudinal:

$$\begin{aligned} L &= |\psi_{in}\rangle\langle\psi_{in}|, \qquad \langle\psi_{in}|\psi_{in}\rangle = 1 \\ L|\psi\rangle &= |\psi_{in}\rangle\langle\psi_{in}|\psi\rangle \\ L^2|\psi\rangle &= |\psi_{in}\rangle\langle\psi_{in}|\psi_{in}\rangle\langle\psi_{in}|\psi\rangle = |\psi_{in}\rangle\langle\psi_{in}|\psi\rangle = L = |\psi\rangle \end{aligned}$$

  (b) Transverse:

$$\begin{aligned} T &= \mathbb{1} - L \\ T^2 &= (\mathbb{1} - L)(\mathbb{1} - L)| = \mathbb{1} - 2L + L^2 = T \\ \langle\psi_{in}|T|\psi\rangle &= \langle\psi_{in}|(\mathbb{1} - |\psi_{in}\rangle\langle\psi_{in}|)|\psi\rangle = 0 \end{aligned}$$

- Separation of the longitudinal and transverse parts of the state:

$$\begin{aligned} |\psi(t)\rangle &= \underbrace{(L + T)}_{\mathbb{1}} e^{-\frac{i}{\hbar}Ht}|\psi_{in}\rangle \\ &= |\psi_{in}\rangle\langle\psi_{in}|e^{-\frac{i}{\hbar}Ht}|\psi_{in}\rangle + Te^{-\frac{i}{\hbar}Ht}|\psi_{in}\rangle \\ &= |\psi_{in}\rangle A(t) + |\phi(t)\rangle, \qquad \langle\psi_{in}|\phi(t)\rangle = 0 \end{aligned}$$

  $|\phi(t)\rangle$: decay product

- Functional equation for the persistence amplitude:

$$\begin{aligned} \langle\psi_{in}|e^{-\frac{i}{\hbar}Ht'}|\psi(t)\rangle &= \langle\psi_{in}|e^{-\frac{i}{\hbar}Ht'}|\psi_{in}\rangle A(t) + \langle\psi_{in}|e^{-\frac{i}{\hbar}Ht'}|\phi(t)\rangle \\ A(t+t') &= A(t)A(t') + \underbrace{\langle\psi_{in}|e^{-\frac{i}{\hbar}Ht'}|\phi(t)\rangle}_{\text{re}-\text{excitation}} \end{aligned}$$

- Without re-excitation: $A(t+t') = A(t)A(t') \Longrightarrow A(t) = A(0)e^{-\frac{t}{\tau}}$

- Evolution of the decay product back to the undecayed state: deviation from the exponential decay

- Irreversibility:

  (a) $H^\dagger = H \Longrightarrow$ there is always a regenerated undecayed state component:

$$\begin{aligned} P^{\pm}_{n\to k} &= \frac{4g^2|H'_{k,n}|^2}{\hbar^2(\omega_{kn} \pm \omega)^2} \sin^2 \frac{1}{2}(\omega_{kn} \pm \omega)t \\ P^{+}_{n\to k} &= P^{-}_{k\to n} \end{aligned}$$

  (b) Irreversibility, non-unitary time evolution is needed to arrive at exponential decays

- Spectral representation:

(a) Spectral function: $H|n\rangle = E_n|n\rangle$

$$
\begin{aligned}
|\psi_{in}\rangle &= \mathbb{1}|\psi_{in}\rangle = \sum_n |n\rangle\langle n|\psi_{in}\rangle \\
A(t) &= \sum_n |\langle n|\psi_{in}\rangle|^2 e^{-\frac{i}{\hbar}E_n t} = \sum_n \int dE |\langle n|\psi_{in}\rangle|^2 \delta(E-E_n) e^{-\frac{i}{\hbar}Et} = \int dE \rho(E) e^{-\frac{i}{\hbar}Et} \\
\rho(E) &= \sum_n |\langle n|\psi_{in}\rangle|^2 \delta(E-E_n).
\end{aligned}
$$

(b) $A(t)$ and $\rho(E)$ are related by Fourier transformation

(c) "Uncertainty relation": the width of $A(t)$ and $\rho(E)$ are inversely proportional

(d) There is no universal decay law

(e) Exponential decay: Lorentzian spectral weight,

$$
\rho(E) = \frac{\Delta E}{\pi[(E-E_1)^2 + \Delta E^2]},
$$

(f) Natural line width of atomic spectra:

i. Partial resummation of the perturbation series of QED

ii. Decay of excited state $\Longrightarrow$ finite life-time $\Longrightarrow$ $E \to E - i\frac{\hbar}{\tau}$, $e^{-\frac{i}{\hbar}Et} \to e^{-\frac{i}{\hbar}(E-i\frac{\hbar}{\tau})t} = e^{-\frac{i}{\hbar}Et} e^{-\frac{t}{\tau}}$

4. **Long time regime:**

(a) Bundedness of the Hamiltonian from below: $\rho(E) = 0$ for $E < E_0$

(b) Shrunk of the support of a Lorentzian spectral function $\rho(E) = 0 \Longrightarrow$ spread of $A(t)$

(c) Slower than exponential decay rate for long time

D. Quantum Zeno-effect

1. **Zeno:** (b. Elea, 488BC) Achilles can not pass a tortoise!

2. **Quantum Zeno effect:** (short time, the parabolic decay regime)

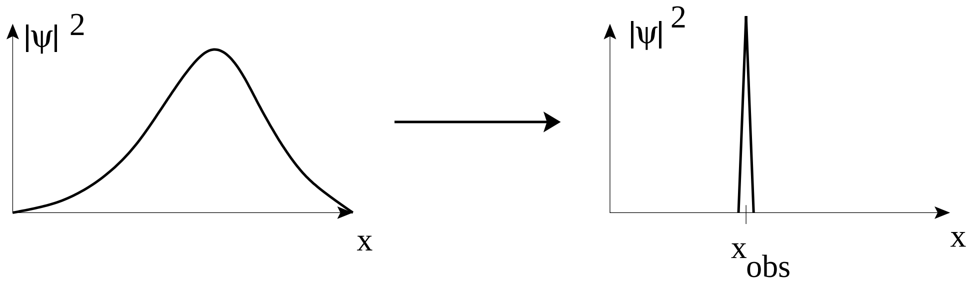
- We observe the system at times $j\Delta t$, $\Delta t = t/n$, $j = 1, \ldots, n$
- Schrödinger equation is local in time $\Longrightarrow$ the eventual decays are independent

$$
\begin{aligned}
i\hbar\partial_t|\psi(t)\rangle &= H|\psi(t)\rangle \\
|\psi((j+1)\Delta t)\rangle &= e^{-\frac{i}{\hbar}\Delta t H}|\psi(j\Delta t)\rangle \\
P_0(t+\Delta t) &= P_0(\Delta t) P_0(t)
\end{aligned}
$$

- Probability of not having decay:

$$\begin{aligned} P_0(t) &= P_0^n(\Delta t) \\ &= \left[1-\left(\frac{t}{nt_Z}\right)^2+\mathcal{O}\left(n^{-3}\right)\right]^n \\ &= e^{n\ln[1-(\frac{t}{nt_Z})^2+\mathcal{O}\left(n^{-3}\right)]} \to 1 \end{aligned}$$

- Continuously monitored radioactive atom does not decay:
  (a) Undecayed state is completely regenerated by the collapse of the wave function (observations)
  (b) Wave function has no time to spread, an $\mathcal{O}\left(\Delta t^2\right)$ effect
  (c) Measurement process:
    - Microscopic $\Longrightarrow$ macroscopic transition (e.g. tracks in Wilsons's could chamber)
    - $A|n\rangle = a_n|n\rangle$, $|\psi\rangle = \sum_n c_n|n\rangle$, $\langle\psi|\psi\rangle = 1$, $\langle\psi|A|\psi\rangle = \sum_n |c_n|^2 a_n$
    - Selection of a spectral element of the observable, a_n
    - QM: averages only
    - No deterministic, causal theory for a single event
    - Hidden parameter theories:
      * Classical description of each microscopical quantity by the help of so far unobserved classical degrees of freedom
      * Non-local $\Longrightarrow$ acausality
      * Contextuel $\Longrightarrow$ no mathematical structure
        · Three observables, A, B and C, $[A,B] = [A,C] = 0$, $[B,C] \neq 0$
        · The value of A depends on whether we measure B or C simultaneously.
    - Collapse of the wave function



    - Non-determinsitic choice of x_{obs}
    - Reality???
    - Quantum Bar Kokhba game

3. Watched pot paradox: (water does not boil in a continuously watched pot)

E. Time-energy uncertainty principle

1. **Heisenberg's uncertainty principle:**

(a) Algebraic derivation:

$$
\begin{aligned}
[A,B] &= iC, \quad A = A^\dagger, \quad B = B^\dagger, \quad C = C^\dagger \\
A_0 &= A - \langle A \rangle, \quad B_0 = B - \langle B \rangle, \quad \langle A \rangle = \begin{cases} \langle \psi | A | \psi \rangle & \textit{pure state} \\ \mathrm{Tr}\rho A & \textit{mixed state} \end{cases}, \quad [A_0, B_0] = iC \\
\Delta A^2 &= \langle A_0^2 \rangle = \langle A^2 \rangle - \langle A \rangle^2, \quad \Delta B^2 = \langle B_0^2 \rangle = \langle B^2 \rangle - \langle B \rangle^2
\end{aligned}
$$

Non-negative norm: $O = A_0 + ixB_0$, $x \in \mathcal{R}$

$$
\begin{aligned}
\langle O O^\dagger \rangle &= \langle A_0^2 \rangle - ix\langle [A_0, B_0] \rangle + x^2 \langle B_0^2 \rangle \geq 0 \\
x_{min} &= -\frac{\langle C \rangle}{2\langle B_0^2 \rangle}
\end{aligned}
$$

Uncertainty:

$$
\boxed{\Delta A \Delta B \geq \frac{1}{2} |\langle C \rangle|}
$$

(b) Fourier transformatiun for x and p: Gaussian wave packet,

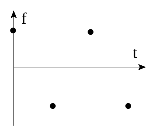
$$
\psi(x) = \int \frac{dk}{2\pi} e^{ikx - \frac{k^2}{2\sigma^2}} = \frac{\sqrt{2\pi}}{\sigma} e^{-\frac{\sigma^2 x^2}{2}}
$$

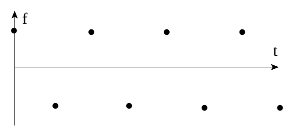
Uncertainty:

$$
\psi(x) = e^{-\frac{x^2}{2\Delta x^2}}, \quad \tilde{\psi}(k) = e^{-\frac{k^2}{2\Delta k^2}} \implies \Delta x \Delta k = 1, \quad \Delta x \Delta p = \hbar
$$

2. **Frequency and observation time:**

(a) Intuitive approach: $T\Delta\omega \approx 1$, $E = \hbar\omega$, $T\Delta E \approx \hbar$



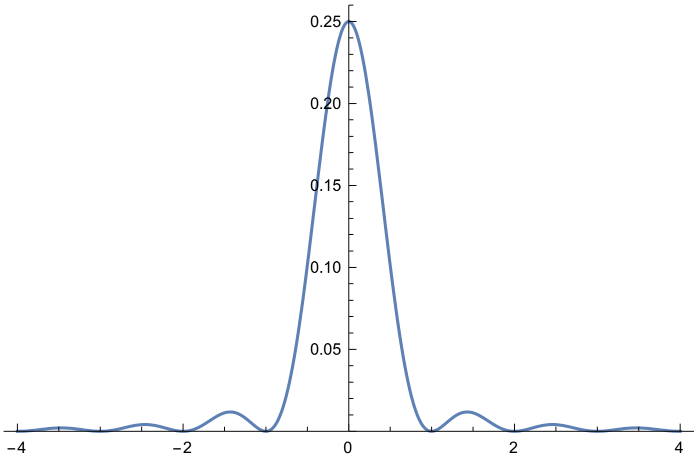


(b) Fourier transformation

(c) Width of the energy spread:

$$
\begin{aligned}
P^{\pm}_{n \to k \neq n} &= \frac{4g^2 |H'_{k,n}|^2}{\hbar^2 (\omega_{kn} \pm \omega)^2} \sin^2 \frac{1}{2} (\omega_{kn} \pm \omega) t \\
t\Delta|\omega \pm \omega_{k,n}| &\approx 2\pi, \quad \Delta E t \approx 2\pi\hbar.
\end{aligned}
$$

$\frac{\sin^2 \pi x}{(2\pi x)^2}$:



F. Fermi's golden rule

- Transition from discrete to continuous spectrum
- Final states are assumed to be decohered (no interference)

$$P_{\text{cont.}\leftarrow\text{discr.}} = \int dE g(E) \frac{|gH_{\text{cont.,discr.}}|^2}{\hbar^2} \frac{4\sin^2\frac{1}{2}(\omega_{\text{cont.,discr.}} \pm \omega)t}{(\omega_{\text{cont.,discr.}} \pm \omega)^2},$$

- Spectral density: $g(E)$ the number of state in the energy interval $[E, E + \Delta E]$
- Change of variable: $E = \hbar\omega \to \beta = \frac{1}{2}(\omega_{\text{cont.,discr.}} \pm \omega)t$, $d\beta = dE\frac{t}{2\hbar}$

$$P_{\text{cont.}\leftarrow\text{discr.}} = \frac{2t}{\hbar}\int d\beta g(E)|gH_{\text{cont.,discr.}}|^2 \frac{\sin^2\beta}{\beta^2}.$$

- Assuming that t is large enough to keep $g(E)$ approximately constant

$$\int_{-\infty}^{\infty} d\beta \frac{\sin^2\beta}{\beta^2} = \pi$$

$$\boxed{P_{\text{cont.}\leftarrow\text{discr.}} \approx t\frac{2\pi}{\hbar} g(E)|gH_{\text{cont.,discr.}}|^2}$$

II. ROTATIONS

A. Finite translations

1. **Classical physics:** coordinate space

$$\boldsymbol{r} \to T(\boldsymbol{a})\boldsymbol{r} = \boldsymbol{r} + \boldsymbol{a}.$$

2. **Functions in space:**

$$f(\boldsymbol{r}) \to f'(\boldsymbol{r}') = f(\boldsymbol{r}' - \boldsymbol{a}).$$

3. **Quantum mechanics:** Hilbert space

$$\psi(\boldsymbol{r}) \to U(T(\boldsymbol{a}))\psi(\boldsymbol{r}) = \psi(\boldsymbol{r}-\boldsymbol{a}).$$

4. **Representation:** $T(\boldsymbol{a}) \to U(T(\boldsymbol{a}))$ preserves the algebraic structure

$$U(T(\boldsymbol{a}))U(T(\boldsymbol{b}))\psi(\boldsymbol{r}) = \psi(\boldsymbol{r}-\boldsymbol{a}-\boldsymbol{b}) = U(T(\boldsymbol{a}+\boldsymbol{b}))\psi(\boldsymbol{r})$$

5. **Unitary representation:**

$$\begin{aligned}\langle\psi|\phi\rangle &= \langle U\psi|U\phi\rangle = \langle\psi|\underbrace{U^\dagger U}_{U^\dagger U=\mathbb{1}}|\phi\rangle \\ \int d\boldsymbol{x}\psi^*(\boldsymbol{x}-\boldsymbol{a})\phi(\boldsymbol{x}-\boldsymbol{a}) &= \int d\boldsymbol{x}\psi^*(\boldsymbol{x})\phi(\boldsymbol{x})\end{aligned}$$

6. **Infinitesimal translations:**

$$\begin{aligned}\boldsymbol{r} &\to \boldsymbol{r}+\delta\boldsymbol{r} \\ \psi(\boldsymbol{r}) &\to \psi(\boldsymbol{r}) - \delta\boldsymbol{r}\boldsymbol{\nabla}\psi(\boldsymbol{r}) = \psi(\boldsymbol{r}) - \frac{i}{\hbar}\delta\boldsymbol{r}\vec{G}\psi(\boldsymbol{r})\end{aligned}$$

Generator: $\vec{G} = \frac{\hbar}{i}\boldsymbol{\nabla} = \boldsymbol{p}$

7. **Finite translations:**

$$\psi(\boldsymbol{r}) \to \psi(\boldsymbol{r}-\boldsymbol{a}) = \sum_{n=0}^{\infty}\frac{(-\boldsymbol{a}\boldsymbol{\nabla})^n}{n!}\psi(\boldsymbol{r}) = e^{-\boldsymbol{a}\boldsymbol{\nabla}}\psi(\boldsymbol{r}) = e^{-\frac{i}{\hbar}\boldsymbol{a}\boldsymbol{p}}\psi(\boldsymbol{r})$$

$$\boxed{U(\boldsymbol{a}) = e^{-\frac{i}{\hbar}\boldsymbol{a}\boldsymbol{p}}}$$

B. Finite rotations

1. **Classical physics:**

- *3x3 matrix:*

$$\boldsymbol{r} \to R_{\boldsymbol{n}}(\alpha)\boldsymbol{r}$$

$\nearrow$ axis $\nwarrow$ angle

- *Orthogonality:*

$$\begin{aligned}(\boldsymbol{u},\boldsymbol{v}) &= \sum_j u_j v_j = (R\boldsymbol{u}, R\boldsymbol{v}) = \sum_j (R\boldsymbol{u})_j, (R\boldsymbol{v})_j = \sum_{jk\ell} R_{jk}u_k R_{j\ell}v_\ell = \sum_{jk\ell} u_k \underbrace{R^{\rm tr}_{kj}R_{j\ell}}_{R^{\rm tr}R=\mathbb{1}} v_\ell \\ R^{-1}_{\boldsymbol{n}}(\alpha) &= R_{\boldsymbol{n}}(-\alpha) = R^{\rm tr}_{\boldsymbol{n}}(\alpha)\end{aligned}$$

- *Rotation around the quantization axis* $\boldsymbol{z}$:

$$R_{\boldsymbol{z}}(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- *Rotation around an arbitrary axis* $\boldsymbol{v} = A\boldsymbol{u}$*:*

$$R_{\boldsymbol{v}}(\alpha) = AR_{\boldsymbol{u}}(\alpha)A^{-1}$$

  Proof:

  - A rotation matrix has a single eigenvactor with zero eigenvalue, the axis, $R_{\boldsymbol{v}}(\alpha)\boldsymbol{v} = \boldsymbol{v}$

$$AR_{\boldsymbol{v}}(\alpha)A^{-1}\boldsymbol{v} = AR_{\boldsymbol{u}}(\alpha)\boldsymbol{u} = A\boldsymbol{u} = \boldsymbol{v}$$

  - The rotation angle remains the same during basis transformation

  In particular: $\boldsymbol{n} = A\boldsymbol{z}$

$$R_{\boldsymbol{n}}(\alpha) = AR_z(\alpha)A^{-1}$$

2. **Functions in space:**

$$f(\boldsymbol{r}) \to U(R)f(\boldsymbol{r}) = f(R^{-1}\boldsymbol{r}).$$

3. **Quantum mechanics:** Hilbert space

$$\psi(\boldsymbol{r}) \to U(R)\psi(\boldsymbol{r}) = \psi(R^{-1}\boldsymbol{r}).$$

4. **Representation:**

$$\begin{aligned} U(R)U(R')\psi(\boldsymbol{r}) &= U(R)\psi(R'^{-1}\boldsymbol{r}) \\ &= \psi(R'^{-1}R^{-1}\boldsymbol{r}) \\ &= \psi((RR')^{-1}\boldsymbol{r}) \\ &= U(RR')\psi(\boldsymbol{r}) \\ U(R)U(R') &= U(RR') \end{aligned}$$

5. **Unitary representation:**

$$\int d\boldsymbol{x}\psi^*(R\boldsymbol{x})\phi(R\boldsymbol{x}) = \int d\boldsymbol{x}\psi^*(\boldsymbol{x})\phi(\boldsymbol{x}) \quad \Longrightarrow \quad U(R)U^\dagger(R) = \mathbb{1}$$

6. **Infinitesimal rotations:**

$$\begin{aligned}\boldsymbol{r} &\rightarrow \boldsymbol{r}+\epsilon\boldsymbol{n}\times\boldsymbol{r}\\ \psi(\boldsymbol{r}) &\rightarrow \psi(\boldsymbol{r})-(\epsilon\boldsymbol{n}\times\boldsymbol{r})\boldsymbol{\nabla}\psi(\boldsymbol{r})=\psi(\boldsymbol{r})-\epsilon\boldsymbol{n}(\boldsymbol{r}\times\boldsymbol{\nabla})\psi(\boldsymbol{r})=\psi(\boldsymbol{r})-\frac{i}{\hbar}\epsilon\boldsymbol{n}\boldsymbol{L}\psi(\boldsymbol{r}),\end{aligned}$$

↗

Generator: angular momentum

7. **Finite rotations:**

   (a) *One dimensional subgroup of rotational around a fixed axis:* $\{R_{\boldsymbol{n}}(\alpha)\}$

   (b) *Generator:* $\boldsymbol{n}\boldsymbol{L}$

   (c) *Representation:*

$$\boxed{U(R_{\boldsymbol{n}}(\alpha))=e^{-\frac{i}{\hbar}\alpha\boldsymbol{n}\boldsymbol{L}}}$$

8. **$\boldsymbol{L}$ is a vector operator:**

   (a) *Definition:* transforms under rotations as a vector and as an operator and the two transformations agree.

   (b) $\boldsymbol{n}_j=A^{-1}\boldsymbol{e}_j$,

$$\begin{aligned}U(R_{\boldsymbol{n}_j}(\alpha)) &= U(A^{-1}R_{\boldsymbol{e}_j}(\alpha)A)\\ &= U(A^{-1})U(R_{\boldsymbol{e}_j}(\alpha))U(A)\\ &= U(A^{-1})e^{-\frac{i}{\hbar}\alpha\boldsymbol{e}_j\boldsymbol{L}}U(A)\\ &= \sum_{n=0}^{\infty}\frac{(-\frac{i}{\hbar}\alpha)^n}{n!}U(A^{-1})(\boldsymbol{e}_j\boldsymbol{L})^nU(A)\\ &= \sum_{n=0}^{\infty}\frac{(-\frac{i}{\hbar}\alpha)^n}{n!}[U(A^{-1})\boldsymbol{e}_j\boldsymbol{L}U(A)]^n\\ &= e^{-\frac{i}{\hbar}\alpha U(A^{-1})\boldsymbol{e}_j\boldsymbol{L}U(A)}.\end{aligned}$$

   (c) *Another expression:* $\boldsymbol{n}_j=A^{-1}\boldsymbol{e}_j=A^{\mathrm{tr}}\boldsymbol{e}_j=\boldsymbol{e}_jA$

$$\begin{aligned}U(R_{\boldsymbol{n}_j}(\alpha)) &= e^{-\frac{i}{\hbar}\alpha\boldsymbol{n}_j\boldsymbol{L}}\\ &= e^{-\frac{i}{\hbar}\alpha\boldsymbol{e}_jA\boldsymbol{L}}\end{aligned}$$

   (d)

$$\boxed{A\boldsymbol{L}=U^{\dagger}(A)\boldsymbol{L}U(A)}$$

↗ ↖

vector operator

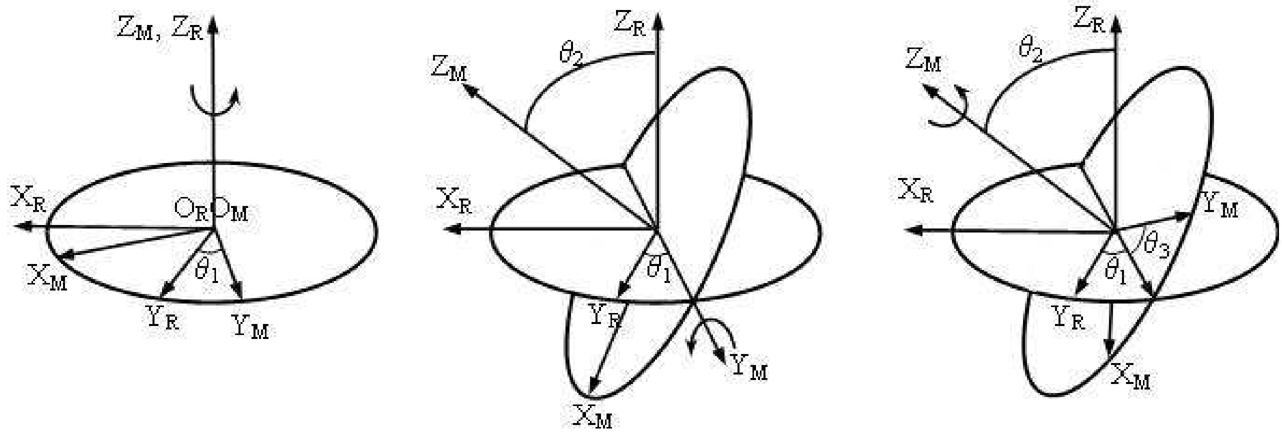
C. Euler angles

1. **Definition:**

$$R(\phi,\theta,\alpha) = R_{\boldsymbol{z}''}(\alpha) R_{\boldsymbol{y}'}(\theta) R_{\boldsymbol{z}}(\phi)$$

$$\nearrow \qquad\qquad \nwarrow$$

$$\boldsymbol{z}'' = R_{\boldsymbol{y}'}(\theta) R_{\boldsymbol{z}}(\phi)\boldsymbol{z} = R_{\boldsymbol{y}'}(\theta)\boldsymbol{z} \qquad\qquad \boldsymbol{y}' = R_{\boldsymbol{z}}(\phi)\boldsymbol{y}$$



$$\theta_1 = \phi \qquad \theta_2 = \theta \qquad \theta_3 = \alpha$$

2. **Another, equivalent expression:** $\boldsymbol{n} = A\boldsymbol{z}$, $R_{\boldsymbol{n}}(\alpha) = A R_z(\alpha) A^{-1}$

$$\begin{aligned}
R_{\boldsymbol{z}''}(\alpha) R_{\boldsymbol{y}'}(\theta) R_{\boldsymbol{z}}(\phi) &= \underbrace{R_{\boldsymbol{y}'}(\theta) R_{\boldsymbol{z}}(\alpha) R_{\boldsymbol{y}'}^{-1}(\theta)}_{R_{\boldsymbol{z}''}(\alpha)} R_{\boldsymbol{y}'}(\theta) R_{\boldsymbol{z}}(\phi) \\
&= \underbrace{R_{\boldsymbol{z}}(\phi) R_{\boldsymbol{y}}(\theta) R_{\boldsymbol{z}}^{-1}(\phi)}_{R_{\boldsymbol{y}'}(\theta)} R_{\boldsymbol{z}}(\alpha) R_{\boldsymbol{z}}(\phi) \\
&= R_{\boldsymbol{z}}(\phi) R_{\boldsymbol{y}}(\theta) R_{\boldsymbol{z}}(\alpha).
\end{aligned}$$

3. **Relation to the parameterization $R_{\boldsymbol{n}}(\alpha)$:** $\boldsymbol{v} = A\boldsymbol{u}$, $R_{\boldsymbol{v}}(\alpha) = A R_{\boldsymbol{u}}(\alpha) A^{-1}$

$$\begin{aligned}
\boldsymbol{n} &= R(\phi,\theta,\chi)\boldsymbol{z} = R_z(\phi) R_y(\theta) R_z(\alpha)\boldsymbol{z} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} \\
R_{\boldsymbol{n}}(\alpha) &= R(\phi,\theta,\chi) R_z(\alpha) R^{-1}(\phi,\theta,\chi)
\end{aligned}$$

Proof:

$$\begin{aligned}
R(\phi,\theta,\chi) R_z(\alpha) R^{-1}(\phi,\theta,\chi) &= R_z(\phi) R_y(\theta) R_z(\chi) R_{\boldsymbol{z}}(\alpha) R_z(-\chi) R_y(-\theta) R_z(-\phi) \\
&= R_z(\phi) R_y(\theta) R_z(\alpha) R_y(-\theta) R_z(-\phi), \quad R_y(\theta)\boldsymbol{z} = \boldsymbol{u} \\
&= R_z(\phi) R_{\boldsymbol{u}}(\alpha) R_z(-\phi), \quad R_z(\phi)\boldsymbol{u} = \boldsymbol{v} \\
&= R_{\boldsymbol{v}}(\alpha), \quad \boldsymbol{n} = R_z(\phi) R_y(\theta)\boldsymbol{z}
\end{aligned}$$

D. Summary of the angular momentum algebra

1. **Orbital angular momentum:**

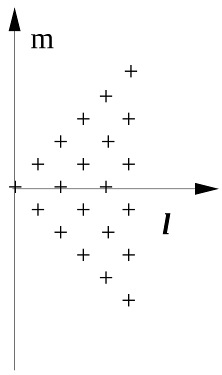
$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$$

2. **Commutation relations:**

$$[L_a, L_b] = i\hbar \sum_c \epsilon_{abc} L_c.$$

3. **Maximal set of commuting operators:** $\{L_z, \boldsymbol{L}^2\} \Longrightarrow$ eigenvalues to label the basis vectors,

$$\begin{aligned} L_z|\ell, m\rangle &= \hbar m|\ell, m\rangle, \quad \boldsymbol{L}^2|\ell, m\rangle = \hbar^2 \ell(\ell+1)|\ell, m\rangle \\ \ell &= 0, 1, \cdots, \quad m \in \{-\ell, -\ell+1, \cdots, \ell-1, \ell\} \end{aligned}$$



4. **Ladder operators:** $L_\pm = L_x \pm iL_y$

$$\begin{aligned} [L_3, L_\pm] &= \pm\hbar L_\pm, \quad [L_+, L_-] = 2\hbar L_z \\ L_\pm|\ell, m\rangle &= \hbar\sqrt{\ell(\ell+1) - m(m \pm 1)}|\ell, m \pm 1\rangle \end{aligned}$$

$$\nearrow$$

to stop at the highers (lowest) state

5. ℓ **remains unvariant under** $\boldsymbol{L}$**:**

$$\langle \ell, m|L_a|\ell', m'\rangle = \delta_{\ell,\ell'} F_a(\ell, m, m')$$

block diagonal structure ℓ

E. Rotational multiplets

1. **Helicity basis:**

$$\begin{aligned}\boldsymbol{u} &= (u_x, u_y, u_z) \to (u_+, u_-, u_z), \quad u_\pm = u_x \pm i u_y \\ \boldsymbol{nL} &= n_x L_x + N_y L_y + n_z L_z \\ &= \frac{1}{2}(n_+ x L_+ + n_- L_-) + n_z L_z = \frac{1}{2}[(n_x - i n_y)(L_x + i L_y) + (n_x + i n_y)(L_x - i L_y)] + n_z L_z\end{aligned}$$

2. **Rotation of $|\ell, m\rangle$:**

$$\begin{aligned}e^{-\frac{i}{\hbar}\alpha \boldsymbol{nL}}|\ell, m\rangle &= \sum_{n=0}^{\infty} \frac{(-\frac{i}{\hbar}\alpha)^n}{n!} (\boldsymbol{nL})^n |\ell, m\rangle \\ &= \sum_{n=0}^{\infty} \frac{(-\frac{i}{\hbar}\alpha)^n}{n!} \left(n_z L_z + \frac{1}{2} n_+ L_- + \frac{1}{2} n_- L_+\right)^n |\ell, m\rangle \\ &= \sum_{-\ell \le m'}^{\ell} c_{m'}(\alpha, \boldsymbol{n}) |\ell, m'\rangle\end{aligned}$$

and all coefficients are non-vanishing if $n_\pm \neq 0$

3. **Rotational multiplet:** $\mathcal{H}_\ell = \{\sum_{m=-\ell}^{\ell} x_m |\ell, m\rangle\}$

4. **Properties:**

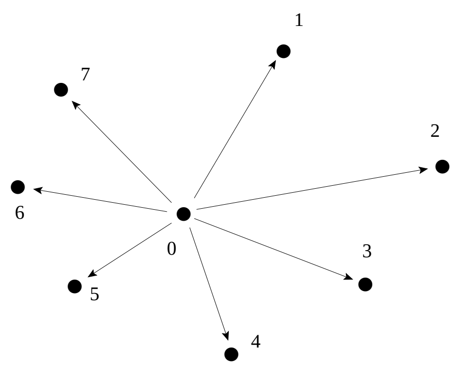
   (a) *Basis:* $\{|\ell, m\rangle | -\ell \le m \le \ell\}$, $\mathrm{Dim}\mathcal{H}_\ell = 2\ell + 1$

   (b) $\mathcal{H}_\ell$ *is closed* with respect to rotations, $e^{-\frac{i}{\hbar}\alpha \boldsymbol{nL}} \mathcal{H}_\ell \subset \mathcal{H}_\ell$.

   (c) $\mathcal{H}_\ell$ *is irreducible* with respect to rotations.

      i. $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ is reducible if each component is closed, $e^{-\frac{i}{\hbar}\alpha \boldsymbol{nL}} \mathcal{H}_j \subset \mathcal{H}_j$, $j = 1, 2$.

      ii. Star condition of irreducibility of $\mathcal{H}$: $\exists |\psi_0\rangle$ such that $\forall |\psi\rangle \in \mathcal{H}$ $\exists R$ such that $\langle \psi | U(R) | \psi_0 \rangle \neq 0$.
      A suitable rotation of $|\psi_0\rangle$ has a projection onto any state.



$$
\begin{aligned}
e^{-\frac{i}{\hbar}\alpha \boldsymbol{n}\boldsymbol{L}}|\ell,m\rangle &= \sum_{n=0}^{\infty}\frac{(-\frac{i}{\hbar}\alpha)^n}{n!}\left(n_zL_z+\frac{1}{2}n_+L_-+\frac{1}{2}n_-L_+\right)^n|\ell,m\rangle \\
\sum_{m'}c^*_{m'}\langle\ell,m'|e^{-\frac{i}{\hbar}\alpha \boldsymbol{n}\boldsymbol{L}}|\ell,m\rangle &= \sum_{n=0}^{\infty}\frac{(-\frac{i}{\hbar}\alpha)^n}{n!}\sum_{m'}c^*_{m'}\langle\ell,m'|\left(n_zL_z+\frac{1}{2}n_+L_-+\frac{1}{2}n_-L_+\right)^n|\ell,m\rangle = 0
\end{aligned}
$$

$\nearrow$

2 equations for 3 variables (not a proof!)

F. Wigner's D matrix

1. D **matrix:** action of rotations within a rotational multiplet

2. **Definition:** $\sum_{\ell',m'}|\ell',m'\rangle\otimes\langle\ell',m'|=\mathbb{1}$

$$
\begin{aligned}
U(R)|\ell,m\rangle &= \mathbb{1}U(R)|\ell,m\rangle = \sum_{\ell',m'}|\ell',m\rangle\langle\ell',m'|U(R)|\ell,m\rangle \\
&= \sum_{m'}|\ell,m\rangle\mathcal{D}^{(\ell)}_{m',m}(R)
\end{aligned}
$$

$$
\boxed{\mathcal{D}^{(\ell)}_{m',m}(R)=\langle\ell',m'|U(R)|\ell,m\rangle}
$$

3. **Euler angles:**

$$
\begin{aligned}
\mathcal{D}^{(\ell)}_{m',m}(R(\alpha,\beta,\gamma)) &= \mathcal{D}^{(\ell)}_{m',m}(R_{\boldsymbol{z}}(\alpha)R_{\boldsymbol{y}}(\beta))R_{\boldsymbol{z}}(\gamma)) \\
&= \sum_{m_1,m_2}\mathcal{D}^{(\ell)}_{m',m_1}(R_{\boldsymbol{z}}(\alpha))\mathcal{D}^{(\ell)}_{m_1,m_2}(R_{\boldsymbol{y}}(\beta))\mathcal{D}^{(\ell)}_{m_2,m}(R_{\boldsymbol{z}}(\gamma)) \\
\langle\ell,m'|e^{-i\frac{\alpha}{\hbar}L_z}|\ell,m\rangle &= = \mathcal{D}^{(\ell)}_{m',m}(R_{\boldsymbol{z}}(\alpha)) = \delta_{m',m}e^{-i\alpha m} \\
\mathcal{D}^{(\ell)}_{m_1,m_2}(R_{\boldsymbol{y}}(\beta)) &= \langle\ell,m'|e^{-i\frac{\beta}{\hbar}L_y}|\ell,m\rangle = d^{(\ell)}_{m_1,m_2}(\beta) \\
\mathcal{D}^{(\ell)}_{m',m}(R(\alpha,\beta,\gamma)) &= e^{-i\alpha m'-i\gamma m}d^{(\ell)}_{m_1,m_2}(\beta)
\end{aligned}
$$

$\nearrow$

Reduced d-matrix

4. **Block diagonal structure:** Basis: $\{\underbrace{|0,0\rangle}_{\mathcal{H}_0}, \underbrace{|1,1\rangle, |0,1\rangle, |-1,1\rangle}_{\mathcal{H}_1}, \underbrace{|2,2\rangle, |1,2\rangle, |0,2\rangle, |-1,2\rangle, |-2,2\rangle}_{\mathcal{H}_2}, \cdots\}$

$$U = \begin{pmatrix} \mathcal{D}^{(0)} & 0 & 0 & \cdots \\ 0 & \mathcal{D}^{(1)} & 0 & \cdots \\ 0 & 0 & \mathcal{D}^{(2)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad \boldsymbol{L} = \begin{pmatrix} \boldsymbol{L}^{(0)} & 0 & 0 & \cdots \\ 0 & \boldsymbol{L}^{(1)} & 0 & \cdots \\ 0 & 0 & \boldsymbol{L}^{(2)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$L_z^{(\ell)} = \hbar \begin{pmatrix} \ell & 0 & \cdots & 0 & 0 \\ 0 & \ell-1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\ell+1 & 0 \\ 0 & 0 & \cdots & 0 & -\ell \end{pmatrix}, \; L_+^{(\ell)} = \hbar \begin{pmatrix} 0 & \sqrt{2\ell} & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \sqrt{2\ell} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \; L_-^{(\ell)} = \hbar \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \sqrt{2\ell} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \sqrt{2\ell} & 0 \end{pmatrix}$$

5. $S = \frac{1}{2}$**:** Pauli matrices,

$$\begin{aligned} \langle \frac{1}{2}, m' | \boldsymbol{L} | \frac{1}{2}, m \rangle &= \frac{\hbar}{2}\boldsymbol{\sigma} = \frac{\hbar}{2}\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \\ \left[\frac{\hbar}{2}\sigma_j, \frac{\hbar}{2}\sigma_k\right] &= \hbar \sum_\ell \epsilon_{jk\ell} \frac{\hbar}{2}\sigma_\ell \quad \Longleftrightarrow \quad [\sigma_j, \sigma_k] = 2\sum_\ell \epsilon_{jk\ell}\sigma_\ell \end{aligned}$$

6. **Two important relations:**

$$\begin{aligned} \sigma_a \sigma_b &= \delta_{a,b} + i \sum_c \epsilon_{abc}\sigma_c \quad \Longleftrightarrow \quad (\boldsymbol{u\sigma}) \cdot (\boldsymbol{v\sigma}) = \mathbb{1}\boldsymbol{uv} + i(\boldsymbol{u} \times \boldsymbol{v})\boldsymbol{\sigma} \\ \sigma_y \boldsymbol{\sigma} \sigma_y &= -\boldsymbol{\sigma}^* \end{aligned}$$

7. **Finite rotation:**

- *Euler's relation:*

$$\begin{aligned} e^{i\alpha} &= 1 + i\alpha + \frac{(i\alpha)^2}{2!} + \frac{(i\alpha)^3}{3!} + \frac{(i\alpha)^4}{4!} + \cdots \\ &= 1 + i\alpha + \frac{(i\alpha)^2}{2!} + \frac{(i\alpha)^3}{3!} + \frac{(i\alpha)^4}{4!} + \cdots \\ &= \frac{1}{2}\left(e^{i\alpha} + e^{-i\alpha}\right) + \frac{1}{2}\left(e^{i\alpha} - e^{-i\alpha}\right) \\ &= \cos\alpha + i\sin\alpha \end{aligned}$$

- *Generalized Euler's relation:*

$$\begin{aligned} e^{i\alpha\boldsymbol{n\sigma}} &= \mathbb{1} + i\alpha\boldsymbol{n\sigma} + \frac{(i\alpha)^2}{2!}(\boldsymbol{n\sigma})^2 + \frac{(i\alpha)^3}{3!}(\boldsymbol{n\sigma})^3 + \frac{(i\alpha)^4}{4!}(\boldsymbol{n\sigma})^4 + \cdots \\ &= \mathbb{1} + i\alpha\boldsymbol{n\sigma} + \mathbb{1}\frac{(i\alpha)^2}{2!}\boldsymbol{n}^2 + \frac{(i\alpha)^3}{3!}\boldsymbol{n}^2\boldsymbol{n\sigma} + \mathbb{1}\frac{(i\alpha)^4}{4!}\boldsymbol{n}^4 + \cdots \\ &= \mathbb{1}\frac{1}{2}\left(e^{i\alpha} + e^{-i\alpha}\right) + \frac{\boldsymbol{n\sigma}}{2}\left(e^{i\alpha} - e^{-i\alpha}\right) \\ &= \mathbb{1}\cos\alpha + i\boldsymbol{n\sigma}\sin\alpha. \end{aligned}$$

- *Reduced d matrix:*

$$
\begin{aligned}
d^{(\frac{1}{2})}_{m',m}(\beta) &= \langle \frac{1}{2}, m'|e^{-i\frac{\beta\sigma_y}{2}}|\frac{1}{2}, m\rangle = \left(\mathbb{1}\cos\frac{\beta}{2} - i\sigma_y \sin\frac{\beta}{2}\right)_{m',m}, \\
d^{(\frac{1}{2})}(\beta) &= \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}.
\end{aligned}
$$

G. Invariant integration over rotations

1. **Separation of the free rotations:**

   (a) *Single particle:* $\psi(\boldsymbol{x}) \to \psi(r, \theta, \phi)$

   (b) *N-particle system:* $\psi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \to \psi(\overbrace{\boldsymbol{X}}^{\text{transl.}}, \overbrace{\phi, \theta, \chi}^{\text{rot.}}, \overbrace{s_1, \ldots, s_{3(N-2)}}^{\text{relative coord.}})$

   $\nearrow$ center of mass; $\nwarrow$ $R(\phi, \theta, \chi)$ connects the laboratory and the body fixed coordinate system

2. **Single particle:**

- *Wave function:*

$$
\chi(\theta, \phi) = \langle \theta, \phi|\psi\rangle = \chi(\boldsymbol{n}(\theta, \phi)), \quad \boldsymbol{n}(\theta, \phi) = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}
$$

- *Resolution of the identity:*

$$
\begin{aligned}
\mathbb{1} &= \int d\Sigma |\theta, \phi\rangle\langle\theta, \phi| = \int d\theta d\phi s_2(\theta, \phi)|\theta, \phi\rangle\langle\theta, \phi| \\
\langle\theta', \phi'|\mathbb{1}|\theta, \phi\rangle &= \int d\theta'' d\phi'' s_2(\theta'', \phi'')\langle\theta', \phi'|\theta'', \phi''\rangle\langle\theta'', \phi''|\theta, \phi\rangle = \langle\theta', \phi'|\theta, \phi\rangle
\end{aligned}
$$

Assume:

$$
\langle\theta', \phi'|\theta, \phi\rangle = f(\theta, \phi)\delta(\theta' - \theta)\delta(\phi' - \phi),
$$

$$
\begin{aligned}
f(\theta, \phi)\delta(\theta' - \theta)\delta(\phi' - \phi) &= \int d\theta'' d\phi'' s_2(\theta'', \phi'') f(\theta'', \phi'')\delta(\theta' - \theta'')\delta(\phi' - \phi'') f(\theta'', \phi'')\delta(\theta'' - \theta)\delta(\phi'' - \phi) \\
&= s_2(\theta, \phi) f^2(\theta, \phi)\delta(\theta' - \theta)\delta(\phi' - \phi) \quad \Longrightarrow \quad f = \frac{1}{s}
\end{aligned}
$$

- *Scalar product:*

$$
\langle\theta', \phi'|\theta, \phi\rangle = \frac{\delta(\theta - \theta')\delta(\phi - \phi')}{s_2(\theta, \phi)}, \quad s_2(\theta, \phi) = \frac{d\Sigma}{d\theta d\phi} = \sin\theta
$$

$$\uparrow$$

surface density of the unit sphere

- *Spherically symmetrical state:* $d\Sigma = d\theta \sin\theta d\phi = d\cos\theta d\phi$

$$\begin{aligned} |0\rangle &= \frac{1}{\sqrt{4\pi}} \int d\Sigma |\boldsymbol{n}(\Sigma)\rangle = \frac{1}{\sqrt{4\pi}} \int d\cos\theta d\phi |\boldsymbol{n}(\theta,\phi)\rangle = \frac{1}{\sqrt{4\pi}} \int d\cos\theta d\phi |\theta,\phi\rangle \\ \chi_0(\theta,\phi) &= \langle \theta,\phi|0\rangle = \frac{1}{\sqrt{4\pi}} \end{aligned}$$

3. N**-particle system:** dynamics over the $SO(3)$ group space

- *Scalar product:*

$$\langle \phi',\theta',\chi'|\phi,\theta,\chi\rangle = \frac{\delta(\phi-\phi')\delta(\theta-\theta')\delta(\chi-\chi')}{s_3(\phi,\theta,\chi)}, \quad s_3(\phi,\theta,\chi) = \frac{dV}{d\phi d\theta d\chi} = \sin\theta$$

- *(Rotational)Invariant integral:* (instead of the resolution of the identity)

$$\int_V d\phi d\theta d\chi s_3(\phi,\theta,\chi) = \int_{R'V} d\phi d\theta d\chi s_3(\phi,\theta,\chi),$$

$$\nearrow$$

V rotated by $R' \in SO(3)$

- *Equivalent form (Haar mesure):* $dR = d\phi d\theta d\chi s_3(\phi,\theta,\chi)$

$$\int dR f(R) = \int d(R'R) f(R) = \int dR f(R'^{-1}R)$$

defined up to a normalization constant

- *Determination of* $s_3(\phi,\theta,\chi)$*:*

(a) Consider the rotational invariant state on the unit sphere:

$$\begin{aligned} |0\rangle &= \int dR U(R)|\boldsymbol{u}\rangle = \int dR |R\boldsymbol{u}\rangle \\ U(R')|\boldsymbol{u}\rangle &= \int dR U(R')U(R)|\boldsymbol{u}\rangle = \int dR U(R'R)|\boldsymbol{u}\rangle = \int dR |R'R\boldsymbol{u}\rangle = |\boldsymbol{u}\rangle \end{aligned}$$

(b) with the choice $\boldsymbol{u} = \boldsymbol{z}$, $R_z(\chi)\boldsymbol{z} = \boldsymbol{z}$

$$|0\rangle = \int d\phi d\theta d\chi s_3(\phi,\theta,\chi) |R(\phi,\theta,0)\boldsymbol{z}\rangle$$

(c) Alternative form: $R(\phi,\theta,\chi)\boldsymbol{z} = \boldsymbol{n}(\phi,\theta)$

$$\begin{aligned} |0\rangle = \int d\phi d\theta d\chi s_3(\phi,\theta,\chi) |\boldsymbol{n}(\phi,\theta)\rangle &= \text{const.} \times \int d\theta d\phi s_2(\theta,\phi) |\boldsymbol{n}(\theta,\phi)\rangle \\ \int d\chi s_3(\phi,\theta,\chi) &= \text{const.} \times s_2(\phi,\theta) = \text{const.} \times \sin\theta \end{aligned}$$

(d) Relating the ϕ- and χ-dependence: $dR = dR^{-1}$, $R^{-1}(\phi,\theta,\chi) = R^{\rm tr}(\phi,\theta,\chi) = R(-\chi, 2\pi-\theta, -\phi)$

No ϕ-dependence $\Longrightarrow$ no χ-dependence

$$s_3(\phi,\theta,\chi) = \sin\theta$$

(e) Haar measure is defined up to const.

(f) Volumes:

$$\begin{aligned}\int_{S_2} d\Sigma &= \int_{-1}^{1} dc \int_{-\pi}^{\pi} d\phi = 4\pi,\\ \int_{SO(3)} dR &= \int_{-1}^{1} dc \int_{-\pi}^{\pi} d\phi \int_{-\pi}^{\pi} d\chi = 8\pi^2\end{aligned}$$

H. Spherical harmonics

1. **Definition:** wave function of $|\ell,m\rangle$,

$$\langle \boldsymbol{n}|\ell,m\rangle = Y^\ell_m(\boldsymbol{n}) = Y^\ell_m(\theta,\phi), \quad \leftarrow \quad \boldsymbol{n} = \begin{pmatrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\theta\end{pmatrix}$$

determined by the structure of the rotation group.

2. **Normalization:**

$$1 = \int_{S_2} d^2n |Y^\ell_m(\boldsymbol{n})|^2,$$

3. **Defininining relations:**

- *Necessary:* and sufficient condition

$$\begin{aligned} L_z Y^\ell_m(\boldsymbol{n}) &= L_z\langle \boldsymbol{n}|\ell,m\rangle = \langle \boldsymbol{n}|L_z|\ell,m\rangle = \hbar m\langle \boldsymbol{n}|\ell,m\rangle = \hbar m Y^\ell_m(\boldsymbol{n}),\\ L_\pm Y^\ell_m(\boldsymbol{n}) &= \langle \boldsymbol{n}|L_\pm|\ell,m\rangle\\ &= \hbar\sqrt{\ell(\ell+1)-m(m\pm1)}\langle \boldsymbol{n}||\ell,m\pm1\rangle\\ &= \hbar\sqrt{\ell(\ell+1)-m(m\pm1)}Y^\ell_{m\pm1}(\boldsymbol{n})\end{aligned}$$

- *Sufficient:*
  - Eigenvectors of hermitian operators $\Longrightarrow$ basis set on the unit sphere
  - Non-degeneracy in L_z: a set of functions on the unit sphere satisfying these eqs. are the spherical harmonics up to a constant

4. **Spherical harmonics in terms of $\mathcal{D}$ matrices:**

- *Relation between the Euler angles and the polar angles:*

$$\begin{aligned} \boldsymbol{n} &= \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} = R(\phi,\theta,\chi)\boldsymbol{z} \\ |\boldsymbol{n}\rangle &= U(R(\phi,\theta,\chi))|\boldsymbol{z}\rangle \end{aligned}$$

  with $\boldsymbol{z} = (0,0,1)$ and χ left arbitrary.

- *Resolution of unity:* $\sum_{\ell',m'} |\ell',m'\rangle \otimes \langle \ell',m'| = \mathbb{1}$

$$|\boldsymbol{n}\rangle = U(R(\phi,\theta,\chi))\mathbb{1}|\boldsymbol{z}\rangle = \sum_{\ell,m} U(R(\phi,\theta,\chi))|\ell,m\rangle \otimes \langle \ell,m|\boldsymbol{z}\rangle$$

- *Projection on* $\langle \ell,m'|$*:*

$$\langle \ell,m'|\boldsymbol{n}\rangle = Y^{\ell *}_{m'}(\boldsymbol{n}) = \sum_m \mathcal{D}^{(\ell)}_{m',m}(R(\phi,\theta,\chi))\langle \ell,m|\boldsymbol{z}\rangle$$

- *Last factor in three steps:*

  (a) Consider

$$\begin{aligned} \langle \ell,m|U(R_{\boldsymbol{z}}(\phi))|\boldsymbol{z}\rangle &= \sum_{\ell',m'} \langle \ell,m|U(R_{\boldsymbol{z}}(\phi))|\ell',m'\rangle\langle \ell',m'|\boldsymbol{z}\rangle \\ &= \sum_{m'} \mathcal{D}^{(\ell)}_{m,m'}(R_{\boldsymbol{z}}(\phi))\langle \ell,m'|\boldsymbol{z}\rangle \\ &= e^{-im\phi}\langle \ell,m|\boldsymbol{z}\rangle \end{aligned}$$

  (b) $\boldsymbol{z} = R_{\boldsymbol{z}}(\phi)\boldsymbol{z} \Longrightarrow$ no ϕ-dependence,

$$\langle \ell,m|U(R_{\boldsymbol{z}}(\phi))|\boldsymbol{z}\rangle = \langle \ell,m|\boldsymbol{z}\rangle$$

  (c) Hence

$$e^{-im\phi}\langle \ell,m|\boldsymbol{z}\rangle = \langle \ell,m|\boldsymbol{z}\rangle$$

  acting on it by $\frac{\partial}{\partial\phi}$ and setting $\phi = 0$:

$$-im\langle \ell,m|\boldsymbol{z}\rangle = 0 \quad \Longrightarrow \quad \langle \ell,m|\boldsymbol{z}\rangle = \delta_{m,0}c_\ell$$

  (d) Normalization:

  - Resolution of unity: $\mathbb{1} = \int_{S_2} d\boldsymbol{n}|\boldsymbol{n}\rangle\langle\boldsymbol{n}|$

– Integration over the unit sphere:

$$\begin{aligned}\int_{S_2} d\Omega f(\boldsymbol{n}) &= \int_{-1}^{1} d\cos\theta \int_{-\pi}^{\pi} d\phi f(\underbrace{\theta,\phi}_{\boldsymbol{n}}), \quad \boldsymbol{n} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} = R(\phi,\theta,\chi)\boldsymbol{z} \\ &= \underbrace{\int_{-1}^{1} d\cos\theta \int_{-\pi}^{\pi} d\phi}_{\int_{S_2} d\boldsymbol{n}}\, f(R(\phi,\theta,\chi)\boldsymbol{z}) \\ &= \frac{1}{2\pi} \underbrace{\int_{-1}^{1} d\cos\theta \int_{-\pi}^{\pi} d\phi \int_{-\pi}^{\pi} d\chi}_{\int_{SO(3)} dR}\, f(R(\phi,\theta,\chi)\boldsymbol{z})\end{aligned}$$

– Normalization:

$$\begin{aligned}1 &= \langle \ell,0|\ell,0\rangle \\ &= \langle \ell,0|\mathbb{1}|\ell,0\rangle \\ &= \int_{S_2} d\boldsymbol{n}\langle \ell,0|\boldsymbol{n}\rangle\langle \boldsymbol{n}|\ell,0\rangle \\ &= \frac{1}{2\pi}\int_{SO(3)} dR\langle \ell,0|U(R)|\boldsymbol{z}\rangle\langle \boldsymbol{z}|U^\dagger(R)|\ell,0\rangle\end{aligned}$$

– Resolution of identity: $\mathbb{1} = \sum_{\ell,m} |\ell,m\rangle\langle\ell,m|$

$$\begin{aligned}1 &= \frac{1}{2\pi}\int_{SO(3)} dR\langle \ell,0|U(R)\mathbb{1}|\boldsymbol{z}\rangle\langle \boldsymbol{z}|\mathbb{1}U^\dagger(R)|\ell,0\rangle \\ &= \frac{1}{2\pi}\sum_{\ell,\ell',m,m'}\int_{SO(3)} dR\langle \ell,0|U(R)|\ell',m'\rangle \underbrace{\langle \ell',m'|\boldsymbol{z}\rangle}_{\delta_{m',0}c_{\ell'}} \underbrace{\langle \boldsymbol{z}|\ell,m\rangle}_{\delta_{m,0}c_\ell}\langle \ell,m|U^\dagger(R)|\ell,0\rangle \\ &= \frac{\langle \ell,0|\boldsymbol{z}\rangle|^2}{2\pi}\underbrace{\int_{SO(3)} dR|\mathcal{D}^{(\ell)}_{0,0}(R)|^2}_{\frac{8\pi^2}{2\ell+1}} \quad\Longrightarrow\quad c_\ell = \sqrt{\frac{2\ell+1}{4\pi}}\end{aligned}$$

(assuming that c_ℓ is real and positive)

- *Finally:*

$$\begin{aligned}Y^\ell_m(\boldsymbol{n}) &= \sum_{m'}\mathcal{D}^{(\ell)*}_{m,m'}(R(\phi,\theta,\chi))\langle \ell,m'|\boldsymbol{z}\rangle^* \\ &= \sqrt{\frac{2\ell+1}{4\pi}}\mathcal{D}^{(\ell)*}_{m,0}(R(\phi,\theta,\chi)) \\ &= \sqrt{\frac{2\ell+1}{4\pi}}e^{im\phi}d^{(\ell)*}_{m,0}(\theta).\end{aligned}$$

5. **Example:** Y^1_m :

- Three functions on the unit sphere, transforming under rotations in an irreducible manner

- $\boldsymbol{n} = (\frac{x}{r}, \frac{y}{r}, \frac{z}{r})$ do the same
- Two different bases for $\mathcal{H}_1$: Y^1_m and $\boldsymbol{n}$
  - Y^1_0: $L_z Y^1_0 = 0$, $L_z z = 0$, normalization: $\int_{S_2} d\boldsymbol{n} |Y(\boldsymbol{n}|^2 = 1$, $Y^1_0 = \sqrt{\frac{3}{4\pi}} \frac{z}{r}$
  - $Y^1_{\pm 1}$:

$$\begin{aligned} Y^1_{\pm 1}(\boldsymbol{n}) &= \frac{1}{\sqrt{2}\hbar} L_\pm Y^1_0(\boldsymbol{n}) \\ &= \frac{1}{\sqrt{2}\hbar} (L_z \pm i L_y) Y^1_0(\boldsymbol{n}) \\ &= \frac{1}{\sqrt{2}\hbar} \sqrt{\frac{3}{4\pi}} [y p_z - z p_y \pm i(z p_x - x p_z)] z, \end{aligned}$$

$\Longrightarrow$

$$\begin{aligned} Y^1_1(\boldsymbol{n}) &= -\sqrt{\frac{3}{8\pi}} \frac{x + iy}{r} = -\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi} \\ Y^1_0(\boldsymbol{n}) &= \sqrt{\frac{3}{4\pi}} \frac{z}{r} = \sqrt{\frac{3}{4\pi}} \cos\theta \\ Y^1_{-1}(\boldsymbol{n}) &= \sqrt{\frac{3}{8\pi}} \frac{x - iy}{r} = \sqrt{\frac{3}{8\pi}} \sin\theta e^{-i\phi}. \end{aligned}$$

III. ADDITION OF ANGULAR MOMENTUM

A. Additive observables and quantum numbers

1. Momentum: Generator of translations, $\boldsymbol{r} \to \boldsymbol{r} + \boldsymbol{\epsilon}$

$$\begin{aligned} \delta\psi(\boldsymbol{r}_1, \boldsymbol{r}_2) &= \psi(\boldsymbol{r}_1 - \boldsymbol{\epsilon}, \boldsymbol{r}_2 - \boldsymbol{\epsilon}) - \psi(\boldsymbol{r}_1, \boldsymbol{r}_2) \\ &= -\frac{i}{\hbar} \boldsymbol{\epsilon} (\boldsymbol{p}_1 + \boldsymbol{p}_2) \psi(\boldsymbol{r}_1, \boldsymbol{r}_2) \\ &= -\frac{i}{\hbar} \boldsymbol{\epsilon} \boldsymbol{P} \psi(\boldsymbol{r}_1, \boldsymbol{r}_2) \quad \Longrightarrow \quad \boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2 \end{aligned}$$

2. Angular momentum: Generator of rotations, $\boldsymbol{r} \to \boldsymbol{r} - \frac{i}{\hbar} \epsilon \boldsymbol{n} \boldsymbol{L}$

- An infinitesimal rotation around the z axis:

$$\begin{aligned} \boldsymbol{r} &= \begin{pmatrix} r\sin\theta\cos\phi \\ r\sin\theta\sin\phi \\ r\cos\theta \end{pmatrix} \to \begin{pmatrix} r\sin\theta\cos(\phi + \epsilon) \\ r\sin\theta\sin(\phi + \epsilon) \\ r\cos\theta \end{pmatrix} \\ \delta\psi(\boldsymbol{r}_1, \boldsymbol{r}_2) &= -\epsilon(\partial_{\phi_1} + \partial_{\phi_2}) \psi(\boldsymbol{r}_1, \boldsymbol{r}_2) \\ &= -\frac{i}{\hbar} \epsilon (L_{1z} + L_{2z}) \psi(\boldsymbol{r}_1, \boldsymbol{r}_2) \\ &= -\frac{i}{\hbar} \epsilon L_z \psi(\boldsymbol{r}_1, \boldsymbol{r}_2) \end{aligned}$$

- General case: $R_{\boldsymbol{n}}(\epsilon)$ is generated by $\boldsymbol{n}(\boldsymbol{L}_1 + \boldsymbol{L}_2)$, $\Longrightarrow \boldsymbol{L} = \boldsymbol{L}_1 + \boldsymbol{L}_2$
- Commutation relations:

$$\begin{aligned} [L_a, L_b] &= [L_{1a} + L_{2a}, L_{1b} + L_{2b}] \\ &= i\hbar \sum_c \epsilon_{a,b,c}(L_{1c} + L_{2c}) \\ &= i\hbar \sum_c \epsilon_{a,b,c} L_c. \end{aligned}$$

- But$\boldsymbol{L}^2 = \boldsymbol{L}_1^2 + \boldsymbol{L}_2^2 + 2\boldsymbol{L}_1\boldsymbol{L}_1$ is not additive $\Longrightarrow \ell$ is not additive neither
- Allowed values of ℓ?
  - Classical mechanics

$$\left(\sqrt{\boldsymbol{L}_1^2} - \sqrt{\boldsymbol{L}_2^2}\right)^2 \le \boldsymbol{L}^2 \le \left(\sqrt{\boldsymbol{L}_1^2} + \sqrt{\boldsymbol{L}_2^2}\right)^2.$$

  - Quantum mechanics?

B. System of two particles

1. System of two particles
   - States $|\phi_1\rangle \in \mathcal{H}_{\ell_1}$, $|\phi_2\rangle \in \mathcal{H}_{\ell_2}$
   - Representation of rotations: $e^{-\frac{i}{\hbar}\alpha \boldsymbol{n}\boldsymbol{L}}|\phi_1\rangle \otimes |\phi_2\rangle$ in $\mathcal{H} = \mathcal{H}_{\ell_1} \otimes \mathcal{H}_{\ell_2}$.
   - Spectrum of $\boldsymbol{L}^2 = (\boldsymbol{L}_1 + \boldsymbol{L}_2)^2$: $\{\ell_1, \ell_2, \dots, \ell_n\} \Longleftrightarrow \mathcal{H} = \mathcal{H}_{\ell_1} \oplus \mathcal{H}_{\ell_2} \oplus \cdots \oplus \mathcal{H}_{\ell_n}$
   - A reducible unitary representation can always be broken up into the direct sum of irreducible representations
2. Two different bases:

   (a) Decoupled basis:

$$|\ell_1, \ell_2, m_1, m_2\rangle = |\ell_1, m_1\rangle \otimes |\ell_2, m_2\rangle, \quad -\ell_j \le m_j \le \ell_j, \quad dim\mathcal{H} = (2\ell_1 + 1)(2\ell_2 + 1)$$

   (b) Coupled basis: $\{|L, M\rangle\}$:

$$\begin{aligned} \boldsymbol{L}^2|L, M\rangle &= \hbar^2 L(L+1)|L, M\rangle, \\ L_3|L, M\rangle &= \hbar M|L, M\rangle, \end{aligned}$$

3. Reduction (construction of the coupled basis):

- $M = M_{max} = m_1 + m_2$:

  (a) $|\ell_1, \ell_2, \ell_1, \ell_2\rangle = |M_{max}, M_{max}\rangle \in \mathcal{H}_{M_{max}} \subset \mathcal{H}$

  (b) $|\ell_1, \ell_2, \ell_1, \ell_2\rangle$ is unique $\Longrightarrow$ no other $\mathcal{H}_{M_{max}} \subset \mathcal{H}$

  (c) No $\mathcal{H}_\ell \subset \mathcal{H}$ with $\ell > M_{max}$

  (d) $U(R)\mathcal{H} \subset \mathcal{H} \Longrightarrow \mathcal{H} = \mathcal{H}_{M_{max}} \oplus \cdots$

- $M = M_{max} - 1 = m_1 + m_2 - 1$:

  (a) Application of $L_- = L_{1-} + L_{2-}$:

$$|M_{max}, M_{max} - 1\rangle = \frac{1}{\hbar\sqrt{2M_{max}}} L_- |M_{max}, M_{max}\rangle.$$

  (b) Two decoupled basis elements with $M = M_{max} - 1$: $|\ell_1, \ell_2, \ell_1 - 1, \ell_2\rangle$ and $|\ell_1, \ell_2, \ell_1, \ell_2 - 1\rangle$

  (c) $S_{M_{max}-1} = \{c_1|\ell_1, \ell_2, \ell_1 - 1, \ell_2\rangle + c_2|\ell_1, \ell_2, \ell_1, \ell_2 - 1\rangle\}$

   – $\dim(S_{M_{max}-1}) = 2$

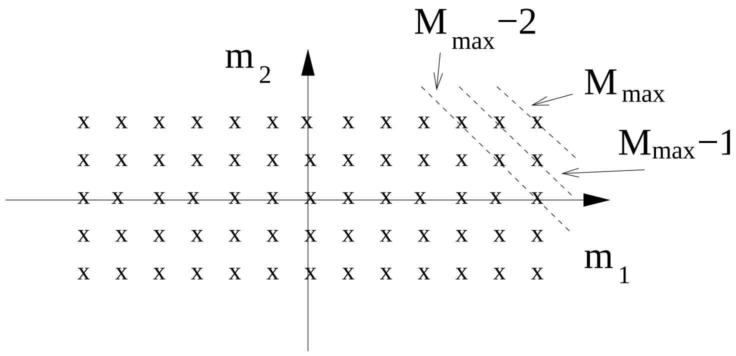
   – $S_{M_{max}-1} \subset \mathcal{H}$

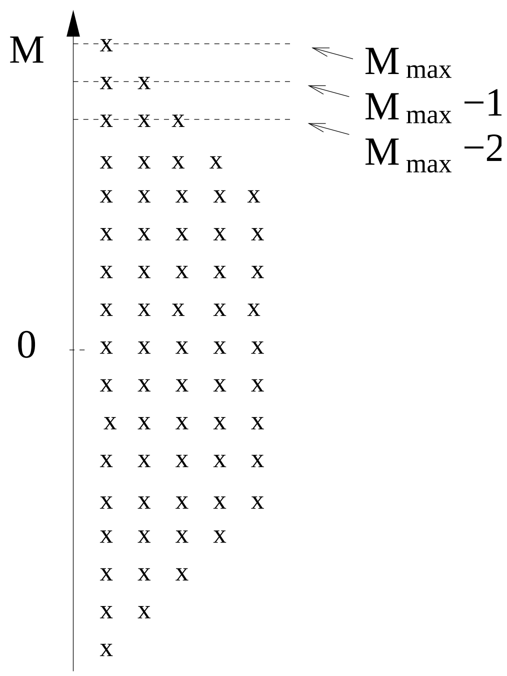
  (d) Choose a basis vector $|M_{max}, M_{max} - 1\rangle \in S_{M_{max}-1}$ such that $|M_{max}, M_{max} - 1 \in\rangle \mathcal{H}_{M_{max}}$

  (e) The other, orthogonal basis vector belongs to a new multiplet, $|M_{max} - 1, M_{max} - 1\rangle \in S_{M_{max}-1}$, $|M_{max} - 1, M_{max} - 1\rangle \in \mathcal{H}_{M_{max}-1}$

  (f) $\in \mathcal{H}_{M_{max}-1} \subset \mathcal{H}$ comes with multiplicity one in $\mathcal{H}$.

  (g) $U(R)\mathcal{H} \subset \mathcal{H} \Longrightarrow \mathcal{H} = \mathcal{H}_{M_{max}} \oplus \mathcal{H}_{M_{max}-1} \oplus \cdots$





- Iteration:

$$\mathcal{H} = \mathcal{H}_{|\ell_1-\ell_2|} \oplus \cdots \oplus \mathcal{H}_{\ell_1+\ell_2}$$

  or

$$\boxed{\ell_1 \otimes \ell_2 = |\ell_1 - \ell_2| \oplus |\ell_1 - \ell_2| + 1 \oplus \cdots \oplus \ell_1 + \ell_2 - 1 \oplus \ell_1 + \ell_2}$$

- Sum rule:

$$\text{dim}\mathcal{H} = (2\ell_1 + 1)(2\ell_1 + 1) = \sum_{|\ell_1-\ell_2|\le\ell\le\ell_1+\ell_2} (2\ell + 1),$$

- Resolution of the identity in $\mathcal{H}$:

$$\mathbb{1} = \sum_{m_1,m_2} |\ell_1, \ell_2, m_1, m_2\rangle\langle\ell_1, \ell_2, m_1, m_2| = \sum_{L,M} |L, M\rangle\langle L, M|$$

- Appears reasonable in the semiclassical limit, $l_1, l_2 \to \infty$

4. Clebsch-Gordan coefficients:

- Definition:

$$\boxed{(\ell_1, \ell_2, m_1, m_2|L, M) = \langle\ell_1, \ell_2, m_1, m_2|L, M\rangle}$$

- Decoupled $\to$ coupled:

$$\begin{aligned} |L, M\rangle &= \sum_{m_1,m_2} |\ell_1, \ell_2, m_1, m_2\rangle\langle\ell_1, \ell_2, m_1, m_2|L, M\rangle \\ &= \sum_{m_1,m_2} |\ell_1, \ell_2, m_1, m_2\rangle(\ell_1, \ell_2, m_1, m_2|L, M) \end{aligned}$$

- Additivity of L_z:

$$(\ell_1, \ell_2, m_1, m_2|L, M) = \delta_{m_1+m_2,M}(\ell_1, \ell_2, m_1, M - m_1|L, M)$$

- Theorem: One can choose the phase of $|\ell, m\rangle$ in such a manner that Clebsch-Gordan coefficients become real.

- Coupled $\to$ decoupled:

$$\begin{aligned} |\ell_1, \ell_2, m_1, m_2\rangle &= \sum_{L,M} |L, M\rangle\langle L, M|\ell_1, \ell_2, m_1, m_2\rangle \\ &= \sum_{L,M} |L, M\rangle\langle\ell_1, \ell_2, m_1, m_2|L, M\rangle^* \\ &= \sum_{L,M} |L, M\rangle(\ell_1, \ell_2, m_1, m_2|L, M)^* \\ &= \sum_{L,M} |L, M\rangle(\ell_1, \ell_2, m_1, m_2|L, M) \end{aligned}$$

5. $\frac{1}{2}\otimes\frac{1}{2}=0\oplus 1$:

- $M=\pm 1$:

$$
\begin{aligned}
|1,\pm 1\rangle &= |\pm\frac{1}{2},\pm\frac{1}{2}\rangle \\
\langle\frac{1}{2},\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2}|1,\pm 1\rangle &= 1
\end{aligned}
$$

- $M=0$:

  (a) $L=1$:

$$
\begin{aligned}
|1,0\rangle &= \frac{1}{\sqrt{2}\hbar}L_-|1,1\rangle \\
&= \frac{1}{2\sqrt{2}}[\sigma_{1x}+\sigma_{2x}-i(\sigma_{1y}+\sigma_{2y})]|\frac{1}{2},\frac{1}{2}\rangle \\
&= \frac{1}{\sqrt{2}}\left[\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}_1+\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}_2\right]|\frac{1}{2},\frac{1}{2}\rangle \\
&= \frac{1}{\sqrt{2}}\left(|\frac{1}{2},-\frac{1}{2}\rangle+|-\frac{1}{2},\frac{1}{2}\rangle\right) \\
\langle\frac{1}{2},\frac{1}{2},\pm\frac{1}{2},\mp\frac{1}{2}|1,0\rangle &= \frac{1}{\sqrt{2}}
\end{aligned}
$$

  (b) $L=0$:

$$
\begin{aligned}
|0,0\rangle &= \frac{1}{\sqrt{2}}\left(|\frac{1}{2},-\frac{1}{2}\rangle-|-\frac{1}{2},\frac{1}{2}\rangle\right) \\
\langle\frac{1}{2},\frac{1}{2},\pm\frac{1}{2},\mp\frac{1}{2}|1,0\rangle &= \pm\frac{1}{\sqrt{2}}
\end{aligned}
$$

  (c) $\mathcal{H}_1$: symmetric with respect to the exchange of the two particles

  $\mathcal{H}_0$: antisymmetric with respect to the exchange of the two particles

IV. SELECTION RULES

A. Tensor operators

1. Definition: $\{T_m^{(\ell)}\}$, $-\ell\le m\le\ell$, transform two equivalent manners

- operators acting in the Hilbert space and
- as tensors, basis vectors of an irreducible multiplet in the linear space of operators,

$$
\boxed{U^\dagger(R)T_m^{(\ell)}U(R)=\sum_{m'}T_{m'}^{(\ell)}\mathcal{D}^{\ell}_{m',m}(R^{-1})}=\sum_{m'}\mathcal{D}^{\ell*}_{m,m'}(R)T_{m'}^{(\ell)}
$$

- $\ell = 1$:

$$A\boldsymbol{L} = U^\dagger(A)\boldsymbol{L}U(A),$$

$$\nearrow \qquad\qquad\qquad \nwarrow$$

vector operator

2. Invariance:

$$\sum_{m'} U^\dagger(R) T^{(\ell)}_{m'} U(R) \mathcal{D}^\ell_{m',m}(R) = T^{(\ell)}_m$$

3. $\ell = 1$:

- $\boldsymbol{r} = (x, y, z)$ as operators (Spherical harmonics: as wave functions)
- Rotation around the z axis:

$$e^{\frac{i}{\hbar}\alpha L_z} T^{(\ell)}_m e^{-\frac{i}{\hbar}\alpha L_z} = \sum_{m'} T^{(\ell)}_{m'} \langle \ell, m' | e^{\frac{i}{\hbar}\alpha L_z} | \ell, m \rangle.$$

- Infinitesimal rotation:

$$\begin{aligned} \left(\mathbb{1} + \frac{i}{\hbar}\alpha L_z\right) T^{(\ell)}_m \left(\mathbb{1} - \frac{i}{\hbar}\alpha L_z\right) &= \sum_{m'} T^{(\ell)}_{m'} \langle \ell, m' | \mathbb{1} + \frac{i}{\hbar}\alpha L_z | \ell, m \rangle \\ T^{(\ell)}_m + \frac{i}{\hbar}\alpha \left[L_z, T^{(\ell)}_m\right] &= \sum_{m'} T^{(\ell)}_{m'} (1 + i\alpha m) \langle \ell, m' | \ell, m \rangle \\ \left[T^{(\ell)}_m, L_z\right] &= -\hbar m T^{(\ell)}_m \end{aligned}$$

- $m = 0$: $R_{\boldsymbol{n}}(\alpha)\boldsymbol{n} = \boldsymbol{n}$

$$T^{(1)}_0 = z,$$

- $m = \pm 1$: vector transformation rules (as wave function)

$$\begin{aligned} Y^1_1(\boldsymbol{n}) &= -\sqrt{\frac{3}{8\pi}}\frac{x + iy}{r}, \quad Y^1_0(\boldsymbol{n}) = \sqrt{\frac{3}{4\pi}}\frac{z}{r}, \quad Y^1_{-1}(\boldsymbol{n}) = \sqrt{\frac{3}{8\pi}}\frac{x - iy}{r} \\ \Longrightarrow \quad T^{(1)}_{\pm 1} &= \mp \frac{x \pm iy}{\sqrt{2}} \end{aligned}$$

- In general: $T^{(\ell)}_m$ from the ℓ-th order multipole expansion

B. Orthogonality relations

1. Orthogonality theorem of representation theory: The set of matrix elements of all irreducible representations of a group form a full, orthogonal basis for functions on the group.

2. Only for $SO(3)$:

$$
\begin{aligned}
\mathcal{D}^{(\ell)}_{m',m}(R(\phi,\theta,\chi) &= \langle \ell, m' | U(R_{\boldsymbol{z}}(\phi) U(R_{\boldsymbol{y}}(\theta) U(R_{\boldsymbol{z}}(\chi) | \ell, m\rangle \\
&= e^{-im'\phi - im\chi} d^{(\ell)}_{m',m}(\theta),
\end{aligned}
$$

is a basis for $SO(3) = \{R(\phi,\theta,\chi)\}$ with the integral measure $d\phi d(\cos\theta) d\chi$,

- Orthogonality:

$$
\boxed{\int dR \mathcal{D}^{(\ell_1)*}_{m'_1,m_1}(R) \mathcal{D}^{(\ell_2)}_{m'_2,m_2}(R) = \frac{8\pi^2}{2\ell_1+1} \delta_{\ell_1,\ell_2} \delta_{m'_1,m'_2} \delta_{m_1,m_2}}
$$

- Completeness:

$$
f(\phi,\theta,\chi) = \sum_{\ell,m,m'} f_{\ell,m,m'} \mathcal{D}^{(\ell)}_{m,m'}(R(\phi,\theta,\chi))
$$

where

$$
f_{\ell,m,m'} = \frac{2\ell_1+1}{8\pi^2} \int_{-\pi}^{\pi} d\phi \int_{-1}^{1} d(\cos\theta) \int_{-\pi}^{\pi} d\chi \mathcal{D}^{(\ell)*}_{m,m'}((\phi,\theta,\chi)) f(\phi,\theta,\chi)
$$

for square integrable functions over $SO(3)$.

- Proof for $\mathcal{D}^{\ell}_{m,m'}(\phi,\theta,\chi) = d^{(\ell)}_{m,m'}(\theta) e^{-im\phi - im'\chi}$:
  - Set of spherical harmonics,

$$
Y^{\ell}_{m}(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}} d^{(\ell)*}_{m,0}(\theta) e^{im\phi}
$$

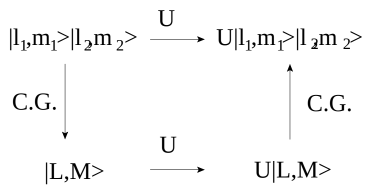
    is a basis over θ, ϕ (S_2) with the integral measure $d\phi d(\cos\theta)$.
  - χ-depenence: different choice of $m' \Longrightarrow \{e^{-im\chi}\}$ is a basis for $S_1 = U(1)$ with the integral measure $d\chi$
  - Normalization:

$$
\int_{SO(3)} dR |\mathcal{D}^{(\ell)}_{0,0}(R)|^2 = \frac{8\pi^2}{2\ell+1}
$$

3. Applied for the addition of angular momentum:

- Clebsch-Gordan coefficient are real $\Longrightarrow$ the basis transformation from the decoupled to the coupled basis is not only unitary but orthogonal,

$$
\begin{aligned}
|L, M\rangle &= \sum_{m_1,m_2} |\ell_1, m_1\rangle \otimes |\ell_2, m_2\rangle (\ell_1, \ell_2, m_1, m_2 | L, M) \\
|\ell_1, m_1\rangle \otimes |\ell_2, m_2\rangle &= \sum_{L,M} |L, M\rangle (\ell_1, \ell_2, m_1, m_2 | L, M)
\end{aligned}
$$



- Two ways of calculating the result of a rotation:

  (a) Commutative diagram:

$$\begin{aligned}
U(R)|\ell_1,m_1\rangle\otimes|\ell_2,m_2\rangle &= \sum_{m_1',m_2'}|\ell_1,m_1'\rangle\otimes|\ell_2,m_2'\rangle\mathcal{D}^{(\ell_1)}_{m_1',m_1}(R)\mathcal{D}^{(\ell_2)}_{m_2',m_2}(R)\\
&= \sum_{L,M,M'}|L,M'\rangle\mathcal{D}^{(L)}_{M',M}(R)(\ell_1,\ell_2,m_1,m_2|L,M)\\
&= \sum_{L,M,M',m_1',m_2'}|\ell_1,m_1'\rangle\otimes|\ell_2,m_2'\rangle(\ell_1,\ell_2,m_1',m_2'|L,M')\\
&\quad\ \mathcal{D}^{(L)}_{M',M}(R)(\ell_1,\ell_2,m_1,m_2|L,M)
\end{aligned}$$

  Projection on $\langle\ell_1,m_1'|\otimes\langle\ell_2,m_2'|$:

$$\langle\ell_1,m_1'|\otimes\langle\ell_2,m_2'|U(R)|\ell_1,m_1\rangle\otimes|\ell_2,m_2\rangle = \mathcal{D}^{(\ell_1)}_{m_1',m_1}(R)\mathcal{D}^{(\ell_2)}_{m_2',m_2}(R)$$
$$= \sum_{L,M,M'}(\ell_1,\ell_2,m_1',m_2'|L,M')\mathcal{D}^{(L)}_{M',M}(R)(\ell_1,\ell_2,m_1,m_2|L,M)$$

  (b) Resolution of the identity:

    i. Trivial (single basis):

$$\begin{aligned}
\mathbb{1} &= \sum_n|n\rangle\langle n|\\
\langle n|A|n'\rangle &= \langle n|\underbrace{\mathbb{1}}_{\sum_m|m\rangle\langle m|}A\underbrace{\mathbb{1}}_{\sum_{m'}|m'\rangle\langle m'|}|n'\rangle = \langle n|A|n'\rangle
\end{aligned}$$

    ii. Less trivial (several bases):

$$\begin{aligned}
\mathbb{1}_d &= \sum_{m_1,m_2}|\ell_1,m_1\rangle\otimes|\ell_2,m_2\rangle\langle\ell_2,m_1|\otimes\langle\ell_2,m_2|\\
\mathbb{1}_c &= \sum_{L,M}|L,M\rangle\langle L,M|\\
\mathbb{1}_dU(R)\mathbb{1}_d &= \mathbb{1}_d\mathbb{1}_cU(R)\mathbb{1}_c\mathbb{1}_d\\
&= \mathbb{1}_d\sum_{L,M,M'}\underbrace{|L,M\rangle\langle L,M|U(R)|L,M'\rangle\langle L,M'|}_{\mathbb{1}_cU(R)\mathbb{1}_c}\mathbb{1}_d
\end{aligned}$$

and

$$
\begin{aligned}
&\langle \ell_1, m_1'| \otimes \langle \ell_2, m_2'|U(R)|\ell_1, m_1\rangle \otimes |\ell_2, m_2\rangle = \\
&= \sum_{L,M,M'} \langle \ell_1, m_1'| \otimes \langle \ell_2, m_2'|L, M\rangle\langle L, M|U(R)|L, M'\rangle\langle L, M'|\ell_1, m_1\rangle \otimes |\ell_2, m_2\rangle \\
&= \sum_{L,M,M'} \langle \ell_1, \ell_2, m_1', m_2'|L, M'\rangle \mathcal{D}^{(L)}_{M',M}(R)\langle L, M|\ell_1, \ell_2, m_1, m_2\rangle \\
&= \sum_{L,M,M'} (\ell_1, \ell_2, m_1', m_2'|L, M')\mathcal{D}^{(L)}_{M',M}(R)(\ell_1, \ell_2, m_1, m_2|L, M)
\end{aligned}
$$

- Orthogonality relation: multiplication by by $\mathcal{D}^{(L)*}_{M',M}(R)$ and integration over R:

$$
\begin{aligned}
&\int dR\mathcal{D}^{(L)*}_{M',M}(R)\underbrace{\langle \ell_1, m_1'| \otimes \langle \ell_2, m_2'|U(R)|\ell_1, m_1\rangle \otimes |\ell_2, m_2\rangle}_{\mathcal{D}^{\ell_1}_{m_1',m_1}(R)\mathcal{D}^{\ell_2}_{m_2',m_2}(R)} \\
&= \sum_{L,M,M'} \int dR\mathcal{D}^{(L)*}_{M',M}(R)(\ell_1, \ell_2, m_1', m_2'|L, M')\mathcal{D}^{(L)}_{M',M}(R)(\ell_1, \ell_2, m_1, m_2|L, M)
\end{aligned}
$$

- Orthogonality relation for Clebsch-Gordan coefficients:

$$
\int dR\mathcal{D}^{(L)*}_{M',M}(R)\mathcal{D}^{(\ell_1)}_{m_1',m_1}(R)\mathcal{D}^{(\ell_2)}_{m_2',m_2}(R) = \frac{8\pi^2}{2L+1}(\ell_1, \ell_2, m_1', m_2'|L, M')(\ell_1, \ell_2, m_1, m_2|L, M)
$$

$\nearrow$ $\langle \ell_1, \ell_2, m_1', m_2'|L, M'\rangle$ $\qquad$ $\nwarrow$ $\langle L, M|\ell_1, \ell_2, m_1, m_2\rangle$

C. Wigner-Eckart theorem

1. Selection rules for a tensor operator: Rotational quantum numbers $\{\ell, m\}$, remaining quantum numbers n

$$
\mathcal{M} = \langle n_1, \ell_1, m_1|T^{(\ell)}_m|n_2, \ell_2, m_2\rangle
$$

$\nearrow$ Rotational quantum numbers $\{\ell, m\}$ $\qquad$ $\nwarrow$ remaining quantum numbers n

2. Derivation of the Wigner-Eckart theorem:

- Tensor operator invariance: $\sum_{m'} U^\dagger(R)T^{(\ell)}_{m'}U(R)\mathcal{D}^\ell_{m',m}(R) = T^{(\ell)}_m$

$$
\begin{aligned}
\mathcal{M} &= \langle n_1, \ell_1, m_1|T^{(\ell)}_m|n_2, \ell_2, m_2\rangle \\
&= \sum_{m'}\langle n_1, \ell_1, m_1|U^\dagger(R)T^{(\ell)}_{m'}U(R)|n_2, \ell_2, m_2\rangle\mathcal{D}^\ell_{m',m}(R) \\
&= \sum_{m'}\langle n_1, \ell_1, m_1|U^\dagger(R)\mathbb{1}T^{(\ell)}_{m'}\mathbb{1}U(R)|n_2, \ell_2, m_2\rangle\mathcal{D}^\ell_{m',m}(R) \qquad \leftarrow \qquad \mathbb{1} = \sum_m |\ell, m\rangle\langle \ell, m| \\
&= \sum_{m_1'm_2'm'} \underbrace{\langle n_1, \ell_1, m_1|U^\dagger(R)|n_1, \ell_1, m_1'\rangle}_{\langle n_1,\ell_1,m_1'|U(R)|n_1,\ell_1,m_1\rangle^*}\langle n_1, \ell_1, m_1'|T^{(\ell)}_{m'}|n_2, \ell_2, m_2'\rangle\langle n_2, \ell_2, m_2'|U(R)|n_2, \ell_2, m_2\rangle \\
&\quad \times\mathcal{D}^\ell_{m',m}(R)
\end{aligned}
$$

Integration over R:

$$\mathcal{M}\int dR = \sum_{m_1',m_2',m'} \langle n_1,\ell_1,m_1'|T^{(\ell)}_{m'}|n_2,\ell_2,m_2'\rangle \int dR\mathcal{D}^{(\ell_1)*}_{m_1',m_1}(R)\mathcal{D}^{(\ell)}_{m',m}(R)\mathcal{D}^{(\ell_2)}_{m_2',m_2}(R),$$

- Orthogonality relation for Clebsch-Gordan coefficients:

$$\mathcal{M}\int dR = \frac{8\pi^2}{2\ell_1+1}(\ell,\ell_2,m,m_2|\ell_1,m_1)\sum_{m_1',m_2',m'}(\ell,\ell_2,m',m_2'|\ell_1,m_1')\langle n_1,\ell_1,m_1'|T^{(\ell)}_{m'}|n_2,\ell_2,m_2'\rangle.$$

Wigner-Eckart:

$$\boxed{\mathcal{M} = (\ell,\ell_2,m,m_2|\ell_1,m_1) \ll n_1,\ell_1|T^{(\ell)}|n_2,\ell_2 \gg}$$

Factorization of the rotational kinematics from the rest of the dynamics

$\nearrow$ $\nwarrow$

$(\ell,\ell_2,m,m_2|\ell_1,m_1)$ reduced matrix element:

$$\ll n_1,\ell_1|T^{(\ell)}|n_2,\ell_2 \gg = \frac{1}{2\ell_1+1}\sum_{m_1',m_2',m'}(\ell,\ell_2,m',m_2'|\ell_1,m_1')\langle n_1,\ell_1,m_1'|T^{(\ell)}_{m'}|n_2,\ell_2,m_2'\rangle,$$

3. Selection rule: $\langle n_1,\ell_1,m_1|T^{(\ell)}_m|n_2,\ell_2,m_2\rangle$ is vanishing if $(\ell,\ell_2,m,m_2|\ell_1,m_1)=0$

4. Examples:

   (a) $\ell = 0$:

$$\begin{aligned}(\ell_2,0,m_2,0|\ell_1,m_1) &= \delta_{\ell_1,\ell_2}\delta_{m_1,m_2}\\ \langle n_1,\ell_1,m_1|T^{(0)}_m|n_2,\ell_2,m_2\rangle &= \delta_{\ell_1,\ell_2}\delta_{m_1,m_2} \ll n_1,\ell_1|T^{(0)}|n_2,\ell_2 \gg\end{aligned}$$

   Rotation invariant potential $U(r) = r^p$

$$\langle n_1,\ell_1,m_1|r^p|n_2,\ell_2,m_2\rangle = \underbrace{\int d\phi\int d(\cos\theta)Y^{\ell_1*}_{m_1}(\theta,\phi)Y^{\ell_2}_{m_2}(\theta,\phi)}_{(\ell_2,0,m_2,0|\ell_1,m_1)}\underbrace{\int drr^{2+p}\eta^*_{n_1,\ell_1}(r)\eta_{n_2,\ell_2}(r)}_{\ll n_1,\ell_1|r^p|n_2,\ell_2\gg}$$

   (b) $\ell = 1$: angular momentum

$$\langle n_1,\ell_1,m_1|T^{(1)}_m|n_2,\ell_2,m_2\rangle = (1,\ell_2,m,m_2|\ell_1,m_1) \ll n_1,\ell_1|T|n_2,\ell_2 \gg$$

   To find the reduced matrix elements for the angular momentum $T^{(1)}_0 = L_z$, $T^{(1)}_{\pm1} = \mp\frac{1}{\sqrt{2}}L_\pm$:

$$\begin{aligned}\langle n_1,\ell_1,\ell_1|L_0|n_2,\ell_2,\ell_2\rangle &= (1,\ell_2,0,\ell_2|\ell_1,\ell_1) \ll n_1,\ell_1|L|n_2,\ell_2 \gg\\ \langle n_1,\ell_1,m|L_0|n_2,\ell_2,m\rangle &= \hbar m\delta_{n_1,n_2}\delta_{\ell_1,\ell_2}\\ (1,\ell_1,0,\ell_1|\ell_1,\ell_1) &= \sqrt{\frac{\ell_1}{\ell_1+1}}\\ \Longrightarrow \quad \ll n_1,\ell_1|L|n_2,\ell_2 \gg &= \delta_{n_1,n_2}\delta_{\ell_1,\ell_2}\hbar\sqrt{\ell(\ell+1)}.\end{aligned}$$