Introduction to General Relativity

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Video: `https://drive.google.com/drive/folders/1Z2TKycQr-itlRpqlgjLqCprDoB6fB7rq?usp=sharing`

Contents

I. FIRST ENCOUNTER

A. Equivalence principle

- **Two kinds of mass:** inertial (m_{in}) and gravitational ($U_{gr} = m_{gr}U$))

  Two kinds of acceleration: dynamical ($\mathbf{a}$), inertial ($\mathbf{a}_{in}$)

$$m_{in}(\mathbf{a} - \mathbf{a}_{in}) = -m_{gr}\nabla U_{gr}(\mathbf{x}),$$

  Lorand Eötvös (1906-09): $m_{in} = m_{gr}$

- **Weak Equivalence Principle:** The world line of a small, free falling body is independent of its composition or structure.

- **Strong Equivalence Principle:** Weak Equivalence Principle valid in the presence of other, non-gravitational force, $\mathbf{F}_{ext}$,

$$m[\mathbf{a} - \mathbf{a}_{in} + \nabla U_{gr}(\mathbf{x})] = \mathbf{F}_{ext}.$$

$$\nearrow \qquad \nearrow$$

- **Common origin** of inertial and gravitational forces?

  1. $U_{gr}(\mathbf{x}) = gz \Longleftrightarrow z \to z - \frac{g}{2}t^2$: weightless state in falling elevator
  2. Homogeneous gravitational force can be eliminated by suitable coordinates
  3. Inhomogeneous gravitational force $\Longrightarrow$ elmination can be made only locally

- **Gravitational force $\Leftrightarrow$ local aspects of the geometry of space and time:**

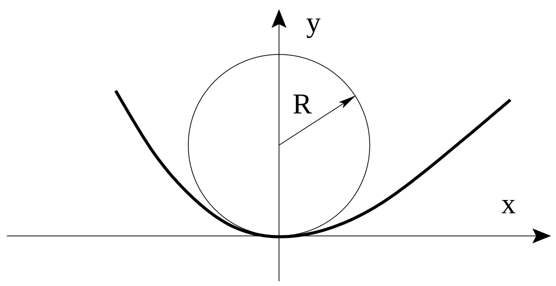
  1. In a small enough region of the space-time it is impossible to detect the presence of a gravitational field and the laws of physics reduce to those of Special Relativity.
  2. The space-time and the physical laws can be made locally Lorentz-invariant.
  3. Gravity as a gauge theory

- **Equivalence principle is violated** by $\mathcal{O}(\hbar)$ quantum effects. e.g. spin

- **Curvature:**

  – The only characteristic classical quantity of the point particle, m, drops out from the gravitational dynamics

  – What determines then the particle trajectory? Ever since Riemann: curvature



– Curvature: radius of the sphere touching the trajectory

$$\begin{aligned}
R^2 &= x^2 + (y-R)^2 = y^2 - 2yR + R^2 + x^2 \\
0 &= y^2 - 2yR + x^2 \\
y &= \frac{1}{2}(2R \pm \sqrt{4R^2 - 4x^2}) \to R - R\sqrt{1 - \frac{x^2}{R^2}} = \frac{x^2}{2R} + \mathcal{O}\left(x^4\right) \\
R &= \frac{1}{\frac{d^2y}{dx^2}} \begin{cases} > 0 & \text{above} \\ < 0 & \text{below} \end{cases}
\end{aligned}$$

– Free-fall:

$$\begin{aligned}
x(t) &= vt, \quad y(t) = y_0 - \frac{g}{2}t^2 \quad \Longrightarrow \quad y(x) = y_0 - \frac{g}{2v^2}x^2 \\
R &= -\frac{v^2}{g} \quad ???
\end{aligned}$$

– Einstein: trajectory in space-time

– Curvature of n-dimensional manifold:

1. Embedd locally into an $n+1$ dimensional Euclidean space
2. Linear approximation: tangent space
3. Quadratic approximation: quadratic form, eigenvalues $\Longrightarrow$ curvature components

B. Comparison with electrodynamics

- **Similarity:**

– Coulomb force: e and $\tilde{m} = m\sqrt{G}$ are coupling constants

$$\mathbf{F}_C = \mathbf{r}\frac{e_1 e_2}{r^3},$$

– Newton's gravitational law:

$$\mathbf{F}_g = -\mathbf{r}G\frac{m_1 m_2}{r^3} = -\mathbf{r}\frac{\tilde{m}_1 \tilde{m}_2}{r^3}$$

- **Differences:**
  - $m > 0$
  - no screening
  - gravity remains long range interaction
  - modifies thermodynamics, e.g. black hole entropy
  - Carriers: $A_\mu(x)$ $(S = 1)$ vs. $g_{\mu\nu}(x)$ $(S = 2 \implies m > 0)$
  - Time-dependen EM and gravitational forces are different
- **Unification:** gauge theory

II. CLASSICAL FIELD THEORIES

1. **Why fields?**

   - *Non-relativistic particles:*

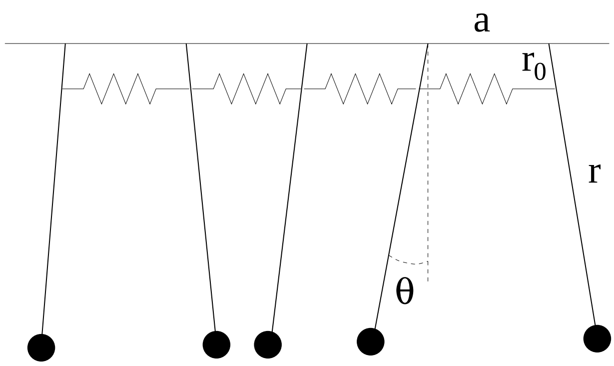
$$\begin{aligned} m_a \frac{d^2\mathbf{x}_a(t)}{dt^2} &= \mathbf{F}_a(\mathbf{x}_1(t), \ldots, \mathbf{x}_N(t)) \\ \mathbf{x}_a(t_i) &= \mathbf{x}_{a,i}, \quad \frac{d\mathbf{x}_a(t)}{dt} = \mathbf{v}_{a,i} \end{aligned}$$

   - *Relativistic particles:* $\dot{x}(s) = \frac{dx}{ds}$

$$\begin{aligned} m_a \ddot{x}^\mu_a(s_a) &= F^\mu_a(x_1(s_1), \ldots, x_N(s_N)) \\ x^\mu_a(s_{a,i}) &= x^\mu_{a,i}, \quad \dot{x}^\mu_a(s_{a,i}) = u^\mu_{a,i}, \quad x^0_a(s_{a,i}) = t_i \end{aligned}$$

   - *Problem:*
     (a) Formal origin:
        i. $\dot{x}^2_a(s) = 1 \implies 0 = \dot{x}_a \ddot{x}_a = \dot{x}_a F_a \implies$ initial coniditions and E.O.M. mixed
        ii. No-go theorem: no covariant function satisfies $0 = \dot{x}_a F_a$
     (b) Physical origin: the instantaneous interaction spreads with infinitely large velocity
   - *Solution:*
     (a) Introduce dynamical degrees of freedom at each space point, $\phi(\mathbf{x})$, a field
     (b) Let the particle interact with the field variable at the same space point
     (c) Self-interaction of the field $\implies$ signal propagates causally $(v_{prop} \leq c)$

2. **Mechanical toy model:** chain of pendulums



A. Variational principle

1. *Single point on the real axis*

1. **Problem:** Identification $x_{cl} \in \mathbb{R}$ in a reparametrization invariant manner

2. **Solution:**

   - Find a function with vanishing derivative at x_{cl} only

   - Impose

$$\frac{df(x)}{dx}_{|x=x_{cl}} = 0$$

   - Reparametrization invariance: $x \to y$,

$$\frac{df(x(y))}{dy}_{|y=y_{cl}} = \underbrace{\frac{df(x)}{dx}_{|x=x_{cl}}}_{0} \frac{dx(y)}{dy}_{|y=y_{cl}} = 0$$

3. **Variational principle:** infinitesimal variation $x \to x + \delta x$,

$$\begin{aligned} f(x_{cl} + \delta x) &= f(x_{cl}) + \delta f(x_{cl}) \\ &= f(x_{cl}) + \delta x \underbrace{f'(x_{cl})}_{0} + \frac{\delta x^2}{2} f''(x_{cl}) + \mathcal{O}\left(\delta x^3\right) \end{aligned}$$

   Variation principle: $(\mathcal{O}\left(\delta x^n\right) = \mathcal{O}\left(\delta y^n\right))$

$$\delta f(x_{cl}) = \mathcal{O}\left(\delta x^2\right),$$

2. *Non-relativistic point particle*

1. **Problem:** Identification of a trajectory in a coordinate choice independent manner

2. **Variational principle:**

- $x_{cl}(t)$ satisfying the auxiliary conditions $x_{cl}(t_i) = x_i$ $x_{cl}(t_f) = x_f$ is to be identified
- Action:

$$S[x] = \int_{t_i}^{t_f} dt L(x(t), \dot{x}(t))$$

$\nearrow$

Lagrangian $L(x(t), \dot{x}(t))$

- Variation: $x(t) \to x(t) + \delta x(t)$, $\delta x(t_i) = \delta x(t_i) = 0$
- E.O.M.:

$$\begin{aligned}
\delta S[x] &= \int_{t_i}^{t_f} dt L\left(x(t) + \delta x(t), \dot{x}(t) + \delta \frac{d}{dt} x(t)\right) - \int_{t_i}^{t_f} dt L(x(t), \dot{x}(t)) \\
&= \int_{t_i}^{t_f} dt \Bigg[L(x(t), \dot{x}(t)) + \delta x(t) \frac{\partial L(x(t), \dot{x}(t))}{\partial x} + \frac{d}{dt} \delta x(t) \frac{\partial L(x(t), \dot{x}(t))}{\partial \dot{x}} + \mathcal{O}\left(\delta x(t)^2\right) \\
&\quad - \int_{t_i}^{t_f} dt L(x(t), \dot{x}(t)) \Bigg] \\
&= \int_{t_i}^{t_f} dt \delta x(t) \left[\frac{\partial L(x(t), \dot{x}(t))}{\partial x} - \frac{d}{dt} \frac{\partial L(x(t), \dot{x}(t))}{\partial \dot{x}} \right] + \underbrace{\delta x(t)}_{0} \frac{\partial L(x(t), \dot{x}(t))}{\partial \dot{x}} \Bigg|_{t_f}^{t_i} + \mathcal{O}\left(\delta x(t)^2\right)
\end{aligned}$$

Euler-Lagrange equation:

$$\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = 0$$

- n-dimensional particle:

$$\boxed{\frac{\partial L}{\partial \mathbf{x}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{x}}} = 0}$$

3. **Choice of the Lagrangian:**

$$L = T - U = \frac{m}{2} \dot{\mathbf{x}}^2 - U(\mathbf{x}) \qquad \Longrightarrow \qquad m \ddot{\mathbf{x}} = -\nabla U(\mathbf{x})$$

4. **Generalized momentum:**

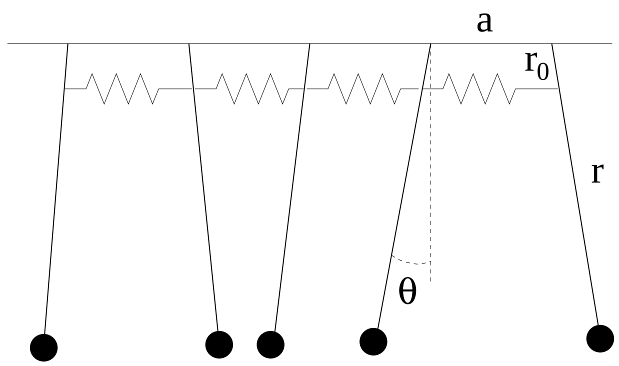
$$p = \frac{\partial L}{\partial \dot{x}} \qquad \Longrightarrow \qquad \dot{p} = \frac{\partial L}{\partial x}$$

5. **Cyclic coordinate:**

$$\frac{\partial L}{\partial x_{cycl}} = 0$$

6. **Conservation law:** The generalized momentum of a cyclic coordinate is conserved

$$\boxed{\dot{p}_{cycl} = 0}$$



7. **Mechanical toy model for field theory:** chain of pendulums

   (a) *Lagrangian:*

$$L = \sum_n \left[\frac{mr^2}{2}\dot{\theta}_n^2 - \frac{kr_0^2}{2}(\theta_{n+1} - \theta_n)^2 - gr\cos\theta_n \right].$$

   (b) *Variable transformation:* $\theta_n(t) \to \Phi\theta_n(t) = \phi(t, x_n)$,

$\Phi = r_0\sqrt{ak}$, $c = a\frac{r_0}{r}\sqrt{\frac{k}{m}}$, $\lambda = \frac{gr}{a}$

$$L = a\sum_n \left[\frac{1}{2c^2}(\partial_t\phi_n)^2 - \frac{1}{2}\left(\frac{\phi_{n+1} - \phi_n}{a}\right)^2 - \lambda\cos\frac{\phi_n}{\Phi} \right]$$

   (c) *Continuum limit:* $a \to 0$

$$L = \int dx \left[\frac{1}{2c^2}(\partial_t\phi(x))^2 - \frac{1}{2}(\partial_x\phi(x))^2 - \lambda\cos\frac{\phi(x)}{\Phi} \right]$$

   (d) *Sine-Gordon model:*

$$S = \int dtdx \left[\frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - \lambda\cos\frac{\phi(x)}{\Phi} \right].$$

3. Scalar field

1. **Problem:** Identification of an n-component field, $\phi_a(x)$, $a = 1, \ldots, n$

2. **Variation:**

$$\phi(x) \to \phi(x) + \delta\phi(x), \qquad \delta\phi(t_i, \mathbf{x}) = \delta\phi(t_f, \mathbf{x}) = 0.$$

3. **Action:**

$$S[\phi] = \int_V dtd^3xL(\phi, \partial\phi)$$

   - Historical convention: dtd^3x rather than $d^4x = dx^0d^3x$

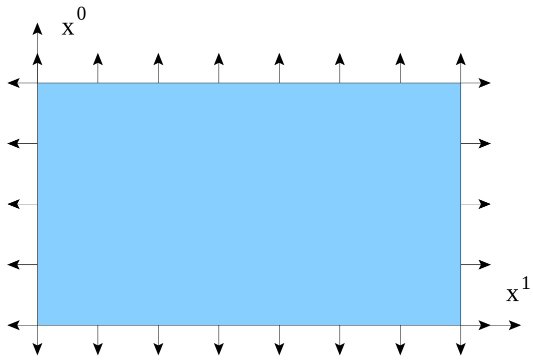
- d^4x will be used below

4. **E.O.M.:**

$$\begin{aligned}\delta S &= \int_V dtd^3x\left(\frac{\partial L(\phi,\partial\phi)}{\partial\phi}\delta\phi+\frac{\partial L(\phi,\partial\phi)}{\partial\partial_\mu\phi}\delta\partial_\mu\phi\right)+\mathcal{O}\left(\delta^2\phi\right)\\ &= \int_V dtd^3x\left(\frac{\partial L(\phi,\partial\phi)}{\partial\phi}\delta\phi+\frac{\partial L(\phi,\partial\phi)}{\partial\partial_\mu\phi}\partial_\mu\delta\phi\right)+\mathcal{O}\left(\delta^2\phi\right)\\ &= \int_{\partial V} ds^\mu\delta\phi\frac{\partial L(\phi,\partial\phi)}{\partial\partial_\mu\phi}+\int_V dtd^3x\delta\phi\left(\frac{\partial L(\phi,\partial\phi)}{\partial\phi}-\partial_\mu\frac{\partial L(\phi,\partial\phi)}{\partial\partial_\mu\phi}\right)+\mathcal{O}\left(\delta^2\phi\right)\end{aligned}$$

$\nwarrow$

0 (local Lagrangian)



Euler-Lagrange equation:

$$\frac{\partial L}{\partial\phi}-\partial_\mu\frac{\partial L}{\partial\partial_\mu\phi}=0$$

5. *N*-**component field:** $\phi_a(x)$, $a=1,\ldots,N$

$$\boxed{\frac{\partial L}{\partial\phi_a}-\partial_\mu\frac{\partial L}{\partial\partial_\mu\phi_a}=0}$$

6. **Current** associated to the field ϕ:

$$j^\mu_\phi=\frac{\partial L}{\partial\partial\phi_\mu}$$

7. **Conservation law:** Current of a cyclic field variable is conserved

$$\boxed{\partial_\mu j^\mu_{\phi_{cycl}}=0}$$

8. **Example:** Scalar field theory ($\hbar=c=1$)

$$L=\frac{1}{2}(\partial\phi)^2-\frac{m^2}{2}\phi^2-U(\phi)\qquad\Longrightarrow\qquad(\partial_\mu\partial^\mu+m^2)=-U'(\phi)$$

B. Noether theorem

1. **Theorem:** There is a conserved current for each continuous symmetry

- *Symmetry:*

$$x^\mu \to x'^\mu, \quad \phi_a(x) \to \phi'_a(x) \quad \Longrightarrow \quad L(\phi, \partial\phi) \to L(\phi', \partial'\phi') + \partial'_\mu \Lambda^\mu$$

- *External and internal spaces:*

$$\phi_a(x) : \underbrace{\mathbb{R}^4}_{\text{external space}} \to \underbrace{\mathbb{R}^n}_{\text{internal space}} .$$

- *Continuous symmetry:* $\exists$ infinitesimal symmetry transformations
  - External symmetry: $x^\mu \to x^\mu + \delta x^\mu$, e.g. Poincare group
  - Internal symmetry: $\phi_a(x) \to \phi_a(x) + \delta\phi_a(x)$, e.g. $\phi(x) \to e^{i\alpha}\phi(x)$ for a complex field
- *Conserved current:* $\partial_\mu j^\mu = 0$, conserved charge: $Q(t)$:

$$\partial_0 Q(t) = \partial_0 \int_V d^3x j^0 = -\int_V d^3x \partial v j = -\int_{\partial V} d\mathbf{s} \cdot \mathbf{j}$$

2. **Continuous groups:**

- $\{\omega(\alpha)\}$:

  (a) continuous topology (infinitesimal neighborhoods)

  (b) multiplication law:

$$\omega(\alpha)\omega(\beta) = \omega(F(\alpha, \beta))$$

  Convention: $\omega(0) = \mathbb{1}$

  (c) Examples: translations, $\alpha = x^\mu$, $F(\alpha, \beta) = \alpha + \beta$

  (d) Ado's theorem: any finite dimensional Lie-algebra is identical with a subspace of the generators of the matrix group $GL(N)$, with sufficiently large N

- *Infinitesimal group elements:*

$$\omega = \mathbb{1} + \sum_{n=1}^{n} \epsilon^a \tau^a + \mathcal{O}\left(\epsilon^2\right),$$

- *Generators:* $\tau^a = \frac{\partial \omega(0)}{\partial \alpha^a}$

- *Exponential map:*

$$\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = \lim_{n\to\infty} e^{n\ln(1+\frac{a}{n})} = \lim_{n\to\infty} e^{n\left(\frac{a}{n}+\mathcal{O}\left(n^{-2}\right)\right)} = e^a$$
$$e^{\sum_a \alpha^a\tau^a} = \lim_{n\to\infty}\left(1+\sum_a \frac{\alpha^a}{n}\tau^a\right)^n$$

Any group element can be obtained in such a form in a connected group

- *Lie-algebra:*

$$[\tau^a,\tau^b] = \sum_c f^{a,b,c}\tau^c.$$

Structure constants: $f^{a,b,c}$, uniquely determine the multiplication of infinitesimal group elements

TABLE I: Real classical matrix groups.

Symbol	Name	Definition	Dimension	Generators
$GL(N)$	general linear group	$\det A \neq 0$[a]	N^2	$\{\tau$: real $N\times N$ matrices$\}$
$SL(N)$	special linear group	$\det A = 1$	N^2-1	$\mathrm{tr}\tau = 0$[b]
$O(N)$	orthogonal group	$A^{tr}A = \mathbb{1}$[c]	$\frac{1}{2}N(N-1)$	$\tau^{tr} = -\tau$
$SO(N)$	special orthogonal group	$A^{tr}A = \mathbb{1}$, $\det A = 1$	$\frac{1}{2}N(N-1)$	$\tau^{tr} = -\tau$, $\mathrm{tr}\tau = 0$

[a]The matrix A is supposed to be an element of the group in question.
[b]$\det(\mathbb{1}+\epsilon\tau) = 1+\epsilon\mathrm{tr}\tau+\mathcal{O}\left(\epsilon^2\right)$
[c]$\det A^{tr}A = (\det A)^2 = 1$ and $\det A = \pm 1$.

3. **Linear internal symmetries:**

- *Linear internal transformation:*

$$\delta x^\mu = 0, \quad \delta\phi_a(x) = \epsilon\tau_{ab}\phi_b(x).$$

- *Symmetry:*

$$L(\phi,\partial\phi) = L(\phi+\epsilon\tau\phi,\partial\phi+\epsilon\tau\partial\phi)+\mathcal{O}\left(\epsilon^2\right).$$

TABLE II: Complex classical matrix groups.

Symbol	Name	Definition	Dimension	Generators
$GL(N,C)$	complex general linear group	$\det A \neq 0$	$2N^2$	$\{\tau$: complex $N\times N$ matrices$\}$
$SL(N,C)$	complex special linear group	$\det A = 1$	$2N^2-2$	$\mathrm{tr}\tau = 0$
$U(N)$	unitary group	$A^\dagger A = \mathbb{1}$[a]	N^2	$\tau^\dagger = -\tau$
$SU(N)$	special unitary group	$A^\dagger A = \mathbb{1}$, $\det A = 1$	N^2-1	$\tau^\dagger = -\tau$, $\mathrm{tr}\tau = 0$

[a]$\det A^\dagger A = (\det A)^*\det A = |\det A|^2 = 1$

- *New field variable:* $\epsilon(x)$, $\phi(x) = \phi_{cl}(x) + \epsilon(x)\tau\phi_{cl}(x)$, $\frac{\delta S[\phi_{cl}]}{\delta\phi} = 0$
- *Linearized Lagrangian* for $\epsilon(x)$ ($\epsilon = 0$ is a solution!):

$$\begin{aligned} L(\epsilon, \partial\epsilon) &= L(\phi_{cl} + \epsilon\tau\phi, \partial\phi_{cl} + \partial\epsilon\tau\phi + \epsilon\tau\partial\phi) \\ &= \frac{\partial L(\phi_{cl}, \partial\phi_{cl})}{\partial\phi}\epsilon\tau + \frac{\partial L(\phi_{cl}, \partial\phi_{cl})}{\partial\partial_\mu\phi}[\partial_\mu\epsilon\tau\phi + \epsilon\tau\partial_\mu\phi] + \mathcal{O}\left(\epsilon^2\right) \end{aligned}$$

- *Symmetry:* $\frac{\partial L}{\partial\phi}\epsilon\tau\phi + \frac{\partial L}{\partial\partial_\mu\phi}\epsilon\tau\partial_\mu\phi = 0 \implies \epsilon$ is a cyclic field $\implies$

$$\begin{aligned} 0 &= \frac{\partial L(\epsilon, \partial\epsilon)}{\partial\epsilon} - \partial_\mu\frac{\partial L(\epsilon, \partial\epsilon)}{\partial\partial_\mu\epsilon} \\ J^\mu_\epsilon &= \frac{\partial L(\epsilon, \partial\epsilon)}{\partial\partial_\mu\epsilon} = \frac{\partial L(\phi_{cl}, \partial\phi_{cl})}{\partial\partial_\mu\phi}\tau\phi \\ \partial_\mu J^\mu_\epsilon &= 0 \end{aligned}$$

  (a) Independent conserved current for each independent direction in the symmetry group

  (b) Defined up to a multiplicative constant

- *Examples:*

  (a) n-component real scalar field: ϕ_a, $a = 1, \cdots, n$, $G = O(n)$,

$$\begin{aligned} L &= \frac{1}{2}(\partial\phi)^2 - V(\phi^2) \\ \delta\phi &= \epsilon^a\tau^a\phi \\ J^a_\mu &= -\partial_\mu\phi\tau^a\phi \end{aligned}$$

  (b) Single complex scalar field: $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$, $G = U(1)$, $\phi(x) \to e^{i\alpha}\phi(x)$

$$\begin{aligned} L &= \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \frac{m^2}{2}(\phi_1^2 + \phi_2^2) - V\left(\frac{1}{2}(\phi_1^2 + \phi_2^2)\right) \\ &= \partial_\mu\phi^*\partial^\mu\phi + \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^\dagger\phi - V(\phi^\dagger\phi) \end{aligned}$$

  Field variable:

  i. $\binom{\phi}{\phi^*}$:

$$\begin{pmatrix}\phi\\ \phi^*\end{pmatrix} : \begin{pmatrix}\phi\\ \phi^*\end{pmatrix} \to \begin{pmatrix}e^{i\alpha}\phi\\ e^{-i\alpha}\phi^*\end{pmatrix}, \quad \delta\begin{pmatrix}\phi\\ \phi^*\end{pmatrix} = i\alpha\begin{pmatrix}\phi\\ -\phi^*\end{pmatrix} = \alpha\tau\begin{pmatrix}\phi\\ \phi^*\end{pmatrix}, \quad \tau = i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$J = -\frac{\partial L}{\partial\partial_\mu\phi}\tau\phi = -i\left(\frac{\partial L}{\partial\partial_\mu\phi}\phi - \frac{\partial L}{\partial\partial_\mu\phi^*}\phi^*\right) = -i(\partial_\mu\phi^*\phi - \phi^*\partial_\mu\phi) = i\phi^*\overleftrightarrow{\partial}_\mu\phi$$

ii. $\binom{\phi_1}{\phi_2}$:

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \ : \ \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \to e^{i\alpha}\phi = \frac{1}{\sqrt{2}}[\cos\alpha\phi_1 - \sin\alpha\phi_2 + i(\cos\alpha\phi_2 + \sin\alpha\phi_1)]$$

$$\delta\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \alpha\begin{pmatrix}-\phi_2\\ \phi_1\end{pmatrix} = \alpha\tau\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \quad \tau = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

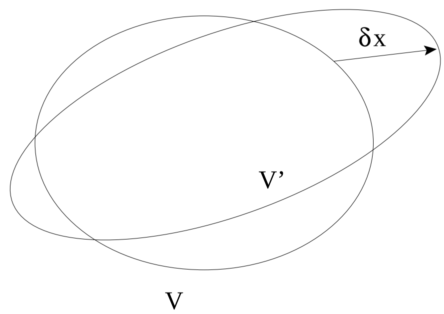
$$\begin{aligned} J &= -\frac{\partial L(\phi, \partial\phi_1)}{\partial\partial_\mu\phi}\tau\phi = -\left(-\frac{\partial L}{\partial\partial_\mu\phi_1}\phi_2 + \frac{\partial L}{\partial\partial_\mu\phi_2}\phi_1\right) = \partial_\mu\phi_1\phi_2 - \partial_\mu\phi_2\phi_1 \\ &= \frac{i}{2}[\partial_\mu(\phi_1 + i\phi_2)^*(\phi_1 + i\phi_2) - (\phi_1 + i\phi_2)^*\partial_\mu(\phi_1 + i\phi_2)] = -i(\partial_\mu\phi^*\phi - \phi^*\partial_\mu\phi) \end{aligned}$$

(c) n-component complex scalar field: ϕ_a, $a = 1, \cdots, n$, $G = U(n)$

$$\begin{aligned} L &= \partial\phi^\dagger\partial\phi - V(\phi^\dagger\phi) \\ \delta\phi &= \epsilon^a\tau^a\phi, \quad \delta\phi^\dagger = \epsilon^a(\phi\tau^a)^\dagger = -\epsilon^a\phi^\dagger\tau^a \\ J^a_\mu &= -\partial_\mu\phi^\dagger\tau^a\phi + \partial_\mu\phi(\tau^a)^{\rm tr}\phi^\dagger = -\partial_\mu\phi^\dagger\tau^a\phi + \phi^\dagger\tau^a\partial_\mu\phi = \phi^\dagger\tau^a\overleftrightarrow{\partial}_\mu\phi \end{aligned}$$

4. **External Symmetries:**

- *Space-time symmetries:*
  (a) Nonrelativistic dynamics: translations of the space-time, rotations of the space and boosts ($4 + 3 + 3 = 10$ dimensional Galilean group)
  (b) Relativistic dynamics: translations and Lorentz transformations of the space-time ($4+3+3 = 10$ dimensional Poincaré group
- *Translations:*
  - Action is rewritten in terms of $x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x)$
  - $x \to x' + \epsilon(x)$, $\phi(x) \to \phi'(x) = \phi(x) + \delta\phi(x)$, $\delta\phi(x) = -\epsilon^\mu\partial_\mu\phi(x)$, $S \to S' = S$



$$\begin{aligned}
0 &= \int_V \delta L(\phi(x), \partial\phi(x)) + \int_{V'-V} dx L(\phi(x), \partial\phi(x)) \\
&= \int_V \delta L(\phi(x), \partial\phi(x)) + \int_{\partial V} dS_\nu \epsilon^\nu L(\phi(x), \partial\phi(x)) \\
&= -\int_V dx \epsilon^\nu \partial_\nu \phi \left(\frac{\partial L(\phi, \partial\phi)}{\partial\phi} - \partial_\mu \frac{\partial L(\phi, \partial\phi)}{\partial\partial_\mu\phi} \right) \quad \leftarrow \quad 0 \quad \text{E.O.M.} \\
& \int_{\partial V} dS_\mu \left[-\epsilon^\nu \partial_\nu \phi \frac{\partial L(\phi, \partial\phi)}{\partial\partial_\mu\phi} + \epsilon^\mu L(\phi, \partial\phi) \right]
\end{aligned}$$

– Rigid shift of the space-time, $\epsilon(x) = \epsilon$:

$$0 = \int_{\partial V} dS_\mu \left[-\partial_\nu \phi \frac{\partial L(\phi, \partial\phi)}{\partial\partial_\mu\phi} + g^\mu_\nu L(\phi, \partial\phi) \right]$$

– V is arbitrary: The energy momentum tensor

$$T^{\mu\nu} = \frac{\partial L}{\partial\partial_\nu\phi} \partial^\mu \phi - g^{\mu\nu} L$$

is conserved

$$\partial_\mu T^{\mu\nu} = 0$$

– "Charge" of the translation ϵ^ν: energy momentum

$$P^\mu = \int d^3x T^{0\mu}$$

– Parameterization:

$$\begin{aligned}
T^{\mu\nu} &= \begin{pmatrix} \epsilon & c\mathbf{p} \\ \frac{1}{c}\mathbf{s} & \sigma \end{pmatrix} \\
\epsilon &= \text{energy density} \\
\mathbf{p} &= \text{momentum density} \\
\mathbf{S} &= \text{energy flux density} \\
\sigma^{jk} &= \text{momentum flux } p^k \text{ in the direction } j
\end{aligned}$$

(c is restored).

- *Lorentz symmetry* $\Longrightarrow$ six conserved currents, angular momentum and generators of Lorentz boosts
- *Conservation of angular momentum* $\Longrightarrow$ $T^{\mu\nu} = T^{\nu\mu}$ for bosonic field theories

III. GAUGE THEORIES

Four fundamental interactions:

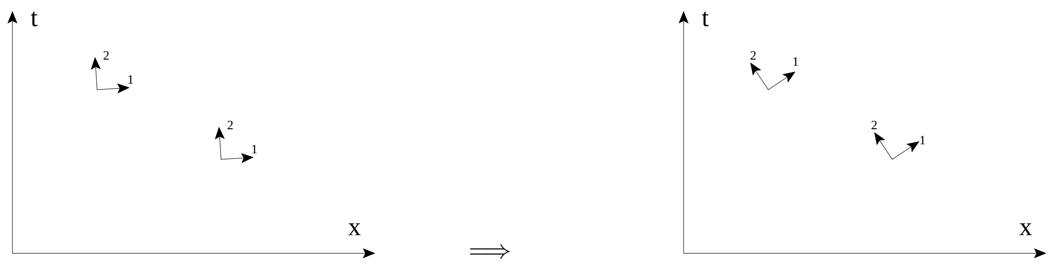
Gravitational, strong, weak, electromagnetic are described by gauge theories

A. Conflict with causality

1. **Global symmetry:** same transformation at each point of the space-time

- *External symmetry:* Poincaré group, $x \to x^\omega = \Lambda x + a$
- *Internal symmetry:* $\omega \in SO(N)$ $(\phi^a \in \mathbb{R}^N)$ or $\omega \in SU(N)$ $(\phi^a \in \mathbb{C}^N)$

$$\phi(x) \to \phi^\omega(x) = \omega\phi(x)$$



2. **Yang and Mills (1954):** global symmetry $\lightning$ causality

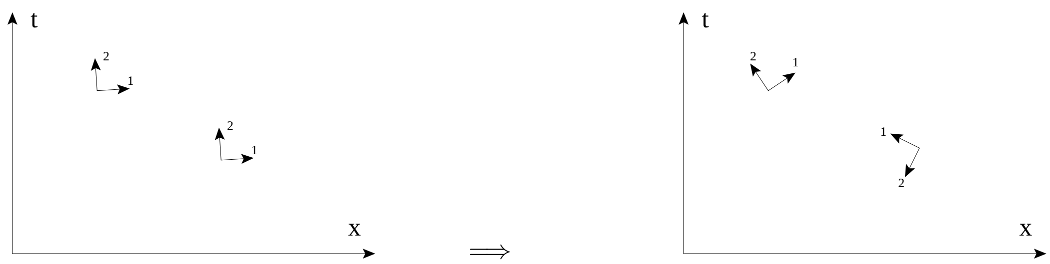
- Symmetry transformation: change of basis
- Global symmetry $\implies$ same change of basis everywhere and always
- Two spatially separated identical experiments compared by using the same basis
- Use of different bases yield inequivalent results
- How can an experiment detect that different basis was used at the other?

3. **Local symmetry:**

$$\psi(x) \to \psi^\omega(x) = \omega(x)\psi(x)$$

$$\nearrow$$

homogeneous transformation rule



4. **Gauge (local) symmetry is a redundancy** of the dynamics:

$$\boxed{\text{(classical) observables are gauge invariant}}$$

5. **Gauging:** Lift a global symmetry to a local one

- Globally symmetrical theory: $\phi^\omega(x) = \omega\phi(x)$

$$L(\phi^\omega, \partial\phi^\omega) = L(\phi, \partial\phi), \quad \omega \in G$$

- Example: N component complex scalar field, ϕ_a, $a = 1, \cdots, N$, $G = U(N)$

$$\begin{aligned} L &= (\partial_\mu\phi)^\dagger\partial^\mu\phi - V(\phi^\dagger\phi) \\ L^\omega &= (\partial_\mu\omega\phi)^\dagger\partial^\mu\omega\phi - V((\omega\phi)^\dagger\omega\phi) \\ &= (\omega\partial_\mu\phi)^\dagger\omega\partial^\mu\phi - V((\omega\phi)^\dagger\omega\phi) = L \end{aligned}$$

- Key equation of global symmetry:

$$\partial_\mu\phi(x) \to \partial_\mu\phi^\omega(x) = \partial_\mu\omega\phi(x) = \omega\partial_\mu\phi(x)$$

$$\nearrow$$

homogeneous transformation rule

- Local symmetry:

$$\partial_\mu\phi(x) \to \partial_\mu\phi^\omega(x) = \partial_\mu\omega(x)\phi(x) = \omega(x)\partial_\mu\phi(x) + \underbrace{(\partial_\mu\omega(x))\phi(x)}_{x-\text{dependend basis change}} \neq \omega(x)\partial_\mu\phi(x)$$

$$\nearrow$$

inhomogeneous transformation rule

- Locally invariant Lagrangian: $L(\phi, \partial\phi) \to L(\phi, D\phi)$

$$\nearrow$$

covariant derivative: homogeneous transformation rule

$$\swarrow$$

$$D_\mu\phi(x) \to D^\omega_\mu\phi^\omega(x) = \omega D_\mu\phi(x)$$

B. Covariant derivative

1. **Key equation** of global symmetry:

$$\partial_\mu\phi(x) \to \partial_\mu\phi^\omega(x) = \partial_\mu\omega\phi(x) = \omega\partial_\mu\phi(x)$$

2. **Local symmetry:**

$$\partial_\mu \phi(x) \to \partial_\mu \phi^\omega(x) = \partial_\mu \omega(x)\phi(x) = \omega(x)\partial_\mu\phi(x) + \underbrace{(\partial_\mu \omega(x))\phi(x)}_{x-\text{dependend basis change}}$$

3. **Partial derivative:**

$$\partial_\mu \phi(x) = \lim_{\epsilon\to 0} \frac{\phi(x+\epsilon n_\mu) - \phi(x)}{\epsilon}$$

compares ϕ in different bases

4. **Change of basis:**

$$\omega(y \leftarrow x) = \mathbb{1} - \Delta x^\mu A_\mu(x) + \mathcal{O}\left(\Delta^2 x\right)$$

$\nearrow$

generators of the gauge group $A_\mu(x) = A^a_\mu(x)\tau^a$

5. **Covariant derivative:**

$$\begin{aligned} D_\mu\phi(x) &= \lim_{\epsilon\to 0} \frac{e^{\epsilon n\cdot A(x+\epsilon n)}\phi(x+\epsilon n_\mu) - \phi(x)}{\epsilon} \\ &= \lim_{\epsilon\to 0} \frac{[1+\epsilon n\cdot A(x+\epsilon n)]\phi(x+\epsilon n_\mu) - \phi(x)}{\epsilon} \\ &= (\partial_\mu + A_\mu)\phi(x) \end{aligned}$$

$$\boxed{D_\mu = \partial_\mu + A_\mu}$$

N.B. Action of A_μ depends on the representation of G: $A_\mu = \sum_a A^a_\mu \tau_a$

e.g. $G = SO(3)$, scalar: $A_\mu = 0$, vector: $A_\mu = \sum_j A^j_\mu S_j$

6. **Transformation of** $A_\mu(x)$ during the gauge transformations:

$$\begin{aligned} \psi(x) &\to \psi^\omega(x) = \omega(x)\psi(x) \quad\Longrightarrow\quad D_\mu\psi \to D^\omega_\mu \psi^\omega \\ D^\omega_\mu\psi^\omega &= (\partial_\mu + A^\omega_\mu)\psi^\omega, \qquad \omega D_\mu\psi = \omega(\partial_\mu + A_\mu)\psi \\ \omega(\partial_\mu + A_\mu)\psi &= (\partial_\mu + A^\omega_\mu)\psi^\omega = (\partial_\mu\omega)\psi + \omega\partial_\mu\psi + A^\omega_\mu\omega\psi \\ A^\omega_\mu &= -\partial_\mu\omega\omega^{-1} + \omega A_\mu \omega^{-1}. \end{aligned}$$

Useful identity:

$$\begin{aligned} \mathbb{1} &= \omega(x)\omega^{-1}(x) \\ 0 &= (\partial_\mu\omega)\omega^{-1} + \omega\partial_\mu\omega^{-1} \end{aligned}$$

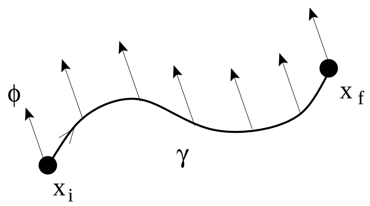
$$\boxed{A_\mu \to A^\omega_\mu = \omega(\partial_\mu + A_\mu)\omega^{-1}}$$

7. **Gauging:**

$$L(\phi, \partial\phi) \to L(\phi, D\phi) = L(\phi, (\partial + A)\phi)$$

C. Parallel transport

1. **Constant vector** $\phi(x)$ along a path $\gamma^\mu : [0,1] \to \mathbb{R}^4$, $\gamma^\mu(0) = x_i^\mu$, $\gamma^\mu(1) = x_f^\mu$



2. **Along the path:**

$$\frac{d\gamma^\mu}{ds} D_\mu \phi(\gamma(s)) = 0$$

3. **End points:**

$$\phi(y) = W_\gamma(y,x)\phi(x)$$

$\nearrow$

path γ dependence?

4. **Evolution equation:**

$$\frac{d\gamma^\mu}{d\tau} D_{y^\mu} W_\gamma(y,x) = 0$$

5. **Infinitesimal path:** $A_\mu(x)$ is constant in the line segment $[x, x+\Delta]$,

$$\begin{aligned}
\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n &= \lim_{n\to\infty} e^{n\ln(1+\frac{a}{n})} = \lim_{n\to\infty} e^{n(\frac{a}{n}+\mathcal{O}(n^{-2}))} = e^a \\
W(x+\Delta x, x) &= \mathbb{1} - \Delta x^\mu A_\mu(x) + \mathcal{O}\left(\Delta x^2\right) \to e^{-\Delta x^\mu A_\mu(x)}
\end{aligned}$$

6. **Homogeneous transformation rule:**

$$\begin{aligned}
\psi^\dagger(y)W_\gamma(y,x)\phi(x) &\to \psi^\dagger(y)W_\gamma(y,x)\phi(x) \\
\left[\phi^\dagger(y)W_\gamma(y,x)\phi(x)\right]^\omega &= \psi^\dagger(y)W_\gamma(y,x)\phi(x) \\
&= \psi^\dagger(y)\omega^\dagger(y)W_\gamma^\omega(y,x)\omega(x)\phi(x) = W_\gamma(y,x) \\
W_\gamma(y,x) &= \omega^\dagger(y)W_\gamma^\omega(y,x)\omega(x) \\
W_\gamma^\omega(y,x) &= \omega(y)W_\gamma(y,x)\omega^\dagger(x)
\end{aligned}$$

$\nearrow$

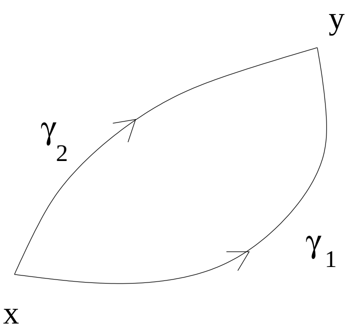
homogeneous transformation rule

7. **Path independent parallel transport** $\Leftrightarrow W_\gamma(x,x) = \mathbb{1}$

(in simply connected space-time) $\Leftrightarrow A_\mu(x)$ is pure gauge ($A^\omega_\mu = \omega(\partial_\mu + \underbrace{A_\mu}_{0})\omega^{-1} = \omega\partial_\mu\omega^{-1}$)

(a) *Multiplication of paths:*

$$\begin{aligned}
\gamma_1(0) &= x,\ \gamma_1(1) = y,\ \gamma_2(0) = y,\ \gamma_2(1) = z \implies \gamma_2\gamma_1(s) = \begin{cases}\gamma_1(2s) & 0 < s < \frac{1}{2} \\ \gamma_2(2s-1) & \frac{1}{2} < s < 1\end{cases} \\
W_{\gamma_2\gamma_1}(z,x) &= W_{\gamma_2}(z,y)W_{\gamma_1}(y,x) \\
\gamma^{-1}(s) &= \gamma(1-s) \\
W_{\gamma^{-1}}(x,y) &= W^{-1}_\gamma(y,z)
\end{aligned}$$



$$W_\gamma(x,x) = W_{\gamma_2}(x,y)W_{\gamma_1}(y,x) = W_{\gamma_2^{-1}}(x,y)W_{\gamma_1}(x,y) = W^{-1}_{\gamma_2}(x,y)W_{\gamma_1}(x,y)$$

(b) *Path independence* $\implies$ $W(x,x) = \mathbb{1}$

(c) $W(x,x) = \mathbb{1}$ $\implies$ *path independence*

(d) *Path independence* $\implies$ $A_\mu(x)$ *is pure gauge*

Reference point x_0: $\omega(x) = W^{-1}(x, x_0)$

$$W'(x_2,x_1) = W^{-1}(x_2,x_0)W(x_2,x_1)W(x_1,x_0) = W(x_0,x_2)W(x_2,x_0) = \mathbb{1} \quad \implies \quad A_\mu(x) = 0$$

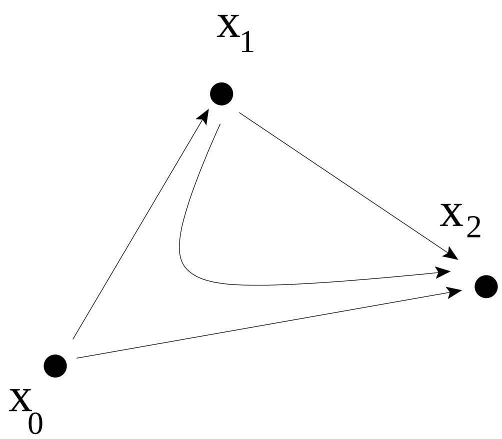
(e) $A_\mu(x)$ *is pure gauge* $\implies$ *Path independence*

$$A^\omega_\mu = \omega\partial_\mu\omega^{-1}, \quad \mathbb{1} = \omega^{-1}(y)W(y,x)\omega(x) \quad \implies \quad W(y,x) = \omega(y)\omega^{-1}(x)$$

D. Field strength tensor

1. **Gauging:** field $\implies$ particle $\implies$ dynamics

$$L(\phi, \partial_\mu\phi) \to L(\phi, (\partial_\mu + A_\mu)\phi) \to L(\phi, D_\mu\phi) + L_A$$



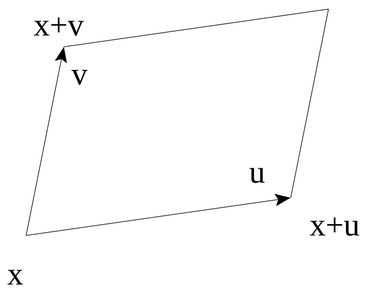
2. L_A:

(a) quadratic in the velocities, $L_A = \mathcal{O}\left((\partial_0 A_\mu)^2\right)$

(b) Lorentz invariant

(c) gauge invariant

3. **Local measure of the path dependence of parallel transport (curvature):**



$$\begin{aligned}
\delta\phi^a &= -F^a_{\ b\mu\nu}u^\mu v^\nu \phi^b \\
U_\Box &= e^{vA(x)}e^{uA(x+v)}e^{-vA(x+u)}e^{-uA(x)} \\
&\approx \left(\mathbb{1} + vA(x) + \frac{1}{2}[vA(x)]^2\right)\left(\mathbb{1} + uA(x+v) + \frac{1}{2}[uA(x+v)]^2\right) \\
&\quad \times \left(\mathbb{1} - vA(x+u) + \frac{1}{2}[vA(x+u)]^2\right)\left(\mathbb{1} - uA(x) + \frac{1}{2}[uA(x)]^2\right) \\
&\approx \mathbb{1} + vA(x) + uA(x+v) - vA(x+u) - uA(x) \\
&\quad + \frac{1}{2}[vA(x)]^2 + \frac{1}{2}[uA(x+v)]^2 + \frac{1}{2}[vA(x+u)]^2 + \frac{1}{2}[uA(x)]^2 \\
&\quad + vA(x)uA(x+v) - vA(x)vA(x+u) - vA(x)uA(x) - uA(x+v)vA(x+u) - uA(x+v)uA(x) \\
&\quad + vA(x+u)uA(x) \\
&\approx \mathbb{1} + (v\partial)uA - (u\partial)vA + \frac{1}{2}(vA)^2 + \frac{1}{2}(uA)^2 + \frac{1}{2}(vA)^2 + \frac{1}{2}(uA)^2 \\
&\quad + vAuA - vAvA - vAuA - uAvA - uAuA + vAuA \\
&\approx \mathbb{1} + (v\partial)uA - (u\partial)vA - (uA)(vA) + (vA)(uA) \\
&= \mathbb{1} - u^\mu v^\nu(\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu])
\end{aligned}$$

Alternative form:

$$\begin{aligned}
[D_\mu, D_\nu]\phi &= [\partial_\mu + A_\mu, \partial_\nu + A_\nu]\phi \\
&= \underbrace{[\partial_\mu, \partial_\nu]}_{0}\phi + \underbrace{[\partial_\mu, A_\nu]\phi}_{\partial_\mu A_\nu\phi - A_\nu\partial_\mu\phi = (\partial_\mu A_\nu)\phi} + \underbrace{[A_\mu, \partial_\nu]\phi}_{-[\partial_\nu, A_\mu]\phi} + [A_\mu, A_\nu]\phi \\
&= (\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu])\phi
\end{aligned}$$

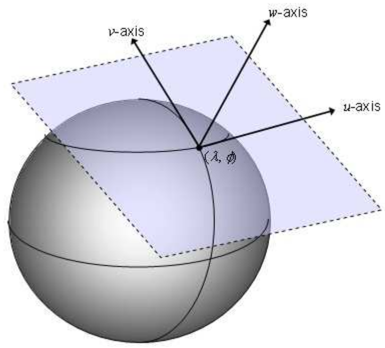
$$\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = [D_\mu, D_\nu]}$$

Generator valued field:

$$\begin{aligned}
F_{\mu\nu} &= F^a_{\mu\nu}\frac{\tau^a}{2i} \\
F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc}A^b_\mu A^c_\nu
\end{aligned}$$

4. **Relation to curvature:** $S_2 = \{\mathbf{n}|\mathbf{n}^2 = 1\}$

   - *Tangent space:* $T_{\mathbf{n}} = \{\dot{\mathbf{x}}(s)_{|s=0}|\mathbf{x}(0) = \mathbf{n}\}$ (velocity space)

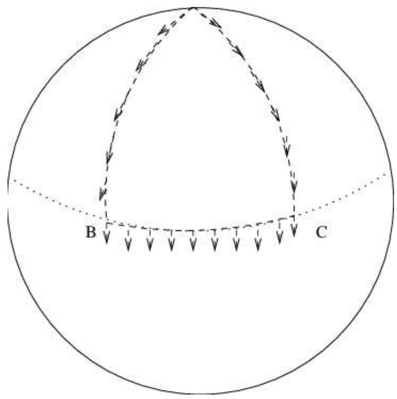


$$L = \mathbf{n} \otimes \mathbf{n}, \quad T = 1\!\!1 - L \quad \leftarrow \quad \text{projection onto the tangent space } T_{\mathbf{n}} = R^2$$

   - *Parallel transport:*

$$\dot{\gamma}^\mu D_\mu u^\phi = 0 \quad \Longleftrightarrow \quad \frac{d}{ds} T u = 0$$

   - *Parallel transport along a spherical triangle:*



5. **Transformation rule under gauge transformations:**

$$\begin{aligned} W_\Box &= 1\!\!1 - u^\mu v^\nu F_{\mu\nu}(x) \to \omega(x)[1\!\!1 - u^\mu v^\nu F_{\mu\nu}(x)]\omega^{-1}(x) \\ F_{\mu\nu}(x) &\to \omega(x) F_{\mu\nu}(x) \omega^{-1}(x) \end{aligned}$$

6. **Pure gauge** ($A_\mu = \omega \partial_\mu \omega^{-1}$) $\Leftrightarrow$ $F = 0$ (connected space-time)

7. **Yang-Mills action:** (unique in space-time with no boundary)

$$L_{YM} = \frac{1}{2g^2} \mathrm{tr}(F_{\mu\nu})^2 = -\frac{1}{4g^2}(F^a_{\mu\nu})^2$$

E. Classical electrodynamics

1. **Action:**

$$\begin{aligned} S &= -c\sum_n m_n\int ds_n - \frac{e}{c}\int dtd^3xj^\mu(x)A_\mu(x) - \frac{1}{16\pi}\int dtd^3xF_{\mu\nu}(x)F^{\mu\nu}(x) \\ &= -c\sum_n m_n\int ds_n - \frac{e}{c^2}\int d^4xj^\mu(x)A_\mu(x) - \frac{1}{16\pi c}\int d^4xF_{\mu\nu}(x)F^{\mu\nu}(x), \end{aligned}$$

- *Electric current:*

$$\begin{aligned} j^\mu(x) &= c\sum_a\int ds\delta(x - x_a(s))\dot{x}^\mu \\ &= c\sum_a\int ds\delta(\mathbf{x} - \mathbf{x}_a(s))\delta(x^0 - x^0_a(s))\dot{x}^\mu \\ &= c\sum_a\delta(\mathbf{x} - \mathbf{x}_a(s))\frac{\dot{x}^\mu}{|\dot{x}^0|} \\ &= \underbrace{\sum_a\delta(\mathbf{x} - \mathbf{x}_a(s))}_{\rho(\mathbf{x})}\frac{dx^\mu}{dt} \\ &= (c\rho, \mathbf{j}) = (c\rho, \rho\mathbf{v}) = \rho\frac{ds}{dt}\dot{x}^\mu \end{aligned}$$

- Current conservation:

$$\begin{aligned} \partial_\mu j^\mu &= \partial_t\rho + \nabla\cdot\mathbf{j} \\ &= \sum_a[-\mathbf{v}_a(t)\nabla\delta(\mathbf{x} - \mathbf{x}_a(t)) + \nabla\delta(\mathbf{x} - \mathbf{x}_a(t))\mathbf{v}_a(t)] = 0 \end{aligned}$$

- *Gauge invariance:* $A_\mu \to A_\mu + \partial_\mu\alpha$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \to \partial_\mu A_\nu + \partial_\mu\partial_\nu\alpha - \partial_\nu A_\mu - \partial_\nu\partial_\mu\alpha = F_{\mu\nu}$

$$\begin{aligned} S &= -c\sum_n m_n\int ds_n - \frac{e}{c^2}\int d^4xj^\mu(x)A_\mu(x) - \frac{1}{16\pi c}\int d^4xF_{\mu\nu}(x)F^{\mu\nu}(x) \\ &\to -c\sum_n m_n\int ds_n - \frac{e}{c^2}\int d^4xj^\mu(x)[A_\mu(x) + \partial_\mu\alpha(x)] - \frac{1}{16\pi c}\int d^4xF_{\mu\nu}(x)F^{\mu\nu}(x) \\ &= -c\sum_n m_n\int ds_n - \frac{e}{c^2}\int d^4x[j^\mu(x)A_\mu(x) - \partial_\mu j^\mu(x)\alpha(x)] - \frac{1}{16\pi c}\int d^4xF_{\mu\nu}(x)F^{\mu\nu}(x) = S \end{aligned}$$

2. **Maxwell's equation:**

$$\begin{aligned} S_A &= -\frac{e}{c^2}\int d^4xj^\mu A_\mu - \frac{1}{16\pi c}\int d^4x(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \\ c\frac{\delta}{\delta A_\nu}: \quad \frac{e}{c}j^\nu &= \frac{1}{4\pi}\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) \\ &= \frac{1}{4\pi}\partial_\mu F^{\mu\nu} \end{aligned}$$

3. **Mechanical E.O.M.:** $x(s) \to x(\tau)$ to avoid $\dot{x}^2(s) = 1$

$$\begin{aligned}
S_{ch} &= -\sum_n \int d\tau \left[m_n c \sqrt{\dot{x}^\mu(\tau) g_{\mu\nu} \dot{x}^\nu(\tau)} + \frac{e}{c} \dot{x}_n^\mu(\tau) A_\mu(x_n(\tau)) \right] \\
\frac{\delta}{\delta x^\mu}: \quad 0 &= -\frac{e}{c} \dot{x}_n^\nu(\tau) \partial_\mu A_\nu(x_n(\tau)) - \frac{d}{d\tau} \left[-mc \frac{\dot{x}_n^\mu(\tau)}{\sqrt{\dot{x}^\mu(\tau) g_{\mu\nu} \dot{x}^\nu(\tau)}} - \frac{e}{c} A_\mu(x_n(\tau)) \right] \\
&= mc \frac{\ddot{x}_n^\mu(\tau)}{\sqrt{\dot{x}^2(\tau)}} - \frac{e}{c} \dot{x}_n^\nu(\tau) [\partial_\mu A_\nu(x_n(\tau)) - \partial_\nu A_\mu(x_n(\tau))] + mc \frac{\dot{x}_n^\mu(\tau)}{[\dot{x}^2(\tau)]^{3/2}} \ddot{x}^\mu(\tau) \dot{x}_\mu(\tau) \\
\tau \to s: \quad mc \ddot{x}_n^\mu(s) &= \frac{e}{c} F_{\mu\nu} \dot{x}_n^\nu(s)
\end{aligned}$$

IV. GRAVITY

A. Classical field theory on curved space-time

1. **Fields over the space-time:** $\phi(x) : E \to I$

   $\nearrow$ $\nwarrow$

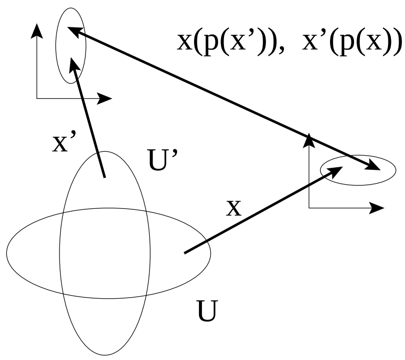
   "where" "what" happens

2. **External space (where):**

   - meter rods, clocks $\Longrightarrow$ coordintes
   - E.O.M. $\Longrightarrow$ coordinate singularities in fields
   - To unfold the coordinate singularities $\Longrightarrow$ local coordinate patchs

     (a) Maps: $\forall\ p \in E\ \exists$ open set $M_p \subset E$, $p \in M_p$

     (b) Coordinates: $\forall\ M\ \exists\ x_M : M \to V \subset \mathcal{R}^d$, $x_M^{-1} = p(x) : V \to M$, $d = 4$ for gravity



     (c) Coordinate transformations: If $p \in M, M'\ \exists\ x(p)$, $x'(p)$, $x'(x) = x'(p(x))$ is infinitely many times differentiable.

3. **Internal space (what):**

   (a) *Non-gravitational events:* $p \to \mathcal{R}^{d_{NG}}$

   (b) *Gravitational events:* directions, vectors, tensors, tangent space $p \to T_p = \mathcal{R}^4$

   T_p : {equivalence classes of world lines$|x(s) \sim x'(s) \ \leftrightarrow x(0) = x'(0) = p, \dot{x}(0) = \dot{x}'(0)$}

   (c) *External space* $\longleftrightarrow$ *Internal space:* movement of free point particle from p: $x^\mu(p, u, s)$

      i. initial conds.: $x^\mu(p, u, 0) = x^\mu(p)\ \dot{x}^\mu(p, u, 0) = u^\mu$

      ii. $X^\mu(u) = x^\mu(p, u, 1) : T_p \to M_p$

      iii. $\exists\ W \subset T_p$, $X^\mu_{|W}$ is invertible, $\exists\ U \subset M_p$, $X^{-1}_{|U} : U \to T_p$

4. **Standard map:**

   (a) *Standard coordinates:* $\bar{x}^\mu(p)$

   (b) *Standard basis in* T_p*:*

$$\bar{e}_\mu = \frac{\partial X^{-1}(p(\bar{x}))}{\partial \bar{x}^\mu}, \qquad e_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad e_{d-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

5. **Change of coordinates:** $x \to x' = x'(x)$

   - *Change of basis in* T_p*:*

     (a) Covariant basis:

$$e_\mu = \frac{\partial X^{-1}(p(x))}{\partial x^\mu} = \frac{\partial X'^{-1}(p(x'))}{\partial x'^\nu}\frac{\partial x'^\nu}{\partial x^\mu} = e'_\nu \frac{\partial x'^\nu}{\partial x^\mu}$$

     (b) Contravariant basis: $e^\mu f_\mu$ remains invariant

$$\begin{aligned} e^\mu &= \frac{\partial x^\mu}{\partial x'^\nu} e'^\nu \\ e^\mu f_\mu &= \frac{\partial x^\mu}{\partial x'^\nu} e'^\nu f'_\rho \frac{\partial x'^\rho}{\partial x^\mu} = \frac{\partial x'^\rho}{\partial x'^\nu} e'^\nu f'_\rho = e'^\nu f'_\nu \end{aligned}$$

     (c) Formal analogy, without physical content:

$$\partial_\mu = \partial'_\nu \frac{\partial x'^\nu}{\partial x^\mu}, \quad \delta x^\mu = \frac{\partial x^\mu}{\partial x'^\nu} \delta x'^\nu$$

$$\boxed{u_\mu = u'_\nu \frac{\partial x'^\nu}{\partial x^\mu}, \quad u^\mu = \frac{\partial x^\mu}{\partial x'^\nu} u'^\nu}$$

B. Geometrcal structure inferred from observations

1. **Metric structure:** measuring distances and time

$$ds^2(x) = dx^\mu g_{\mu\nu}(x) dx^\nu$$

   Equivalence principle: signature $+, -, -, -$

2. **Affine structure:** Locally (equivalence principle) conserved vectors of free particles (velocity, angular momentum)

$$\dot{\gamma}^\mu D_\mu \phi = \dot{\gamma}^\mu (\partial_\mu + \Gamma_\mu)\phi = 0$$

3. **Torsion:** $\mathcal{O}(\hbar)$ spin effect, ignored

C. Gauge group

1. **Internal space of gravity:**

   - *What happens?* ballistic motion, $I = \{\text{particles pass the observer}\}$
   - *Gauge theory* with $I_p = T_p$

$$\boxed{\text{Gravity: } E \text{ and } I \text{ are related}}$$

2. **Space-time diffeomorphism:**

   - *Special relativity:*
     - Translation invariance $\Longrightarrow$ relative coordinates
     - Boost invariance $\Longrightarrow$ relative velocity
   - *General relativity:*
     - Space-time diffeomorphism invariance $\Longrightarrow$ relative higher order derivatives

3. **Space-time diffeomorphism as gauge symmetry:**

   - *Coordinate tetrad vectors:* $e_\mu(x)$ instead of the coordinates $x^\mu(p)$
     - Holonomic tetrads:

$$\boxed{\{e_\mu(x)\} \text{ is compatible with a coordinate system} \Longleftrightarrow \partial_\mu e_\nu = \partial_\nu e_\mu}$$

       Necessary: $\partial_\mu e_\nu = \partial_\mu \partial_\nu X^{-1}(p(x)) = \partial_\nu \partial_\mu X^{-1}(p(x)) = \partial_\nu e_\mu$

       Sufficient: local existence and unicity of integral curves $\partial_\mu X'^{-1}(p(x)) = e_\mu(x)$

     - Diffeomorphism preserves holonomy

$$e'_\mu = e_\nu \frac{\partial x^\nu}{\partial x'^\mu} \qquad \Longrightarrow \qquad \partial'_\kappa e'_\mu = e_\nu \frac{\partial^2 x^\nu}{\partial x'^\mu \partial x'^\kappa} = \partial'_\mu e'_\kappa$$

   - *Coordinate transformation:* $x^\mu \to x'^\mu(x) \quad \Longleftrightarrow \quad e'_\mu = e_\nu \frac{\partial x^\nu}{\partial x'^\mu}$
   - *Gauge group:* $G = GL(4)$
     - Gauge transformation:

$$e_\mu(x) \to e'_\mu(x) = \omega_\mu{}^\nu(x) e_\nu(x), \quad \omega_\mu{}^\nu = \frac{\partial x^\nu}{\partial x'^\mu}, \quad \det[\omega] \neq 0$$

     - Gauge field: $(A_\mu)^\nu{}_\rho = \Gamma^\nu_{\rho\mu}$ affine connection
   - *Internal space:* Tangent space T_x (relating external and internal spaces)
   - *Non-gauge field:* metric tensor, $g_{\mu\nu}$

4. **Poincaré group as gauge symmetry:**

- *Equivalence Principle:* local Lorentz frames Which one to use? A freedom of the Poincaré group
- *Two bases in the internal space:* $\dot{x}^\mu \in T_x^{world}$, $\dot{\xi}^a \in T_p^{Lorentz}$

$$\dot{x}^\mu = \dot{\xi}^a e_a^\mu, \quad \dot{x}^a = \dot{\xi}^\mu e_\mu^a$$

- *Local Lorentz transformations:*

$$e^a(x) \to e'^a(x) = \omega^a{}_b(x) e^b(x).$$

- *Local translations:* space-time diffeomorphism

$$x^\mu(x) \to x'^\mu(x) = x^\mu(x) + \delta x^\mu(x), \quad \xi^a(x) \to \xi'^a(x) = \xi^a(x) + \delta\xi^a(x), \quad \delta\xi^a(x) = e_\mu^a \delta x^\mu(x).$$

- *Fermions:* need Lorentz basis

D. Gauge theory of diffeomorphism

1. **Covariant derivative:**

- *Gauge field:* affin connection, $(\Gamma_\rho)^\mu{}_\nu = \Gamma^\mu{}_{\nu\rho}$

  (a) Covariant field:

$$\begin{aligned} D_\nu v^\mu &= \partial_\nu v^\mu + \Gamma^\mu{}_{\rho\nu} v^\rho \\ (D_\nu v)^\mu &= (\partial_\nu v + \Gamma_\nu v)^\mu \end{aligned}$$

  (b) *Leibnitz's rule:*

$$\begin{aligned} D(uv) &= (Du)v + u(Dv) \\ \text{e.g. } D_\mu(fu^\mu) &= (D_\mu f)u^\mu + (fD_\mu u^\mu) = \partial_\mu f u^\mu + fD_\mu u^\mu \end{aligned}$$

$\uparrow$ scalar representation of $GL(4)$

  (c) *Contravariant field:*

$$\begin{aligned} D_\mu(u^\nu v_\nu) &= (\partial_\mu u + \Gamma_\mu u)^\nu v_\nu + u^\nu(\partial_\mu v - v\Gamma_\mu)_\nu = \partial_\mu(u^\nu v_\nu) \\ (D_\nu v)_\mu &= (\partial_\nu v - v\Gamma_\nu)_\mu \end{aligned}$$

(d) *Mixed tensor field:* acting on each index, eg.

$$D_\nu v^\mu_\rho = \partial_\nu v^\mu_\rho + \Gamma^\mu_{\kappa\nu} v^\kappa_\rho - v^\mu_\kappa \Gamma^\kappa_{\rho\nu}.$$

It is easy to check that such an extension reproduces Leibniz rule,

$$D_\mu(u^\rho v^\sigma) = (D_\mu u^\rho)v^\sigma + u^\rho D_\mu(v^\sigma).$$

- **Gauge transformation:**

(a) $A_\mu \to A'_\mu = \omega(\partial_\mu + A_\mu)\omega^{-1}$

$$\begin{aligned} \delta x^\mu \to \delta x'^\mu &= \omega^\mu{}_\kappa \delta x^\kappa, \quad \omega^\mu{}_\kappa = \frac{\partial x'^\mu}{\partial x^\kappa}, \quad (\omega^{-1})^\mu{}_\kappa = \frac{\partial x^\mu}{\partial x'^\kappa} \\ \Gamma^\mu{}_{\nu\rho} \to \Gamma'^\mu{}_{\nu\rho} &= \frac{\partial x^\sigma}{\partial x'^\rho} \omega^\mu{}_\kappa (\delta^\kappa_\lambda \partial_\sigma + \Gamma^\kappa{}_{\lambda\sigma})(\omega^{-1})^\lambda{}_\nu \end{aligned}$$

$\nearrow$

transformation of a covariant index Γ_ρ: $\partial_\rho \to \partial'_\rho = \frac{\partial x^\sigma}{\partial x'^\rho}\partial_\sigma$

$$\begin{aligned} \Gamma'^\mu{}_{\nu\rho} &= \frac{\partial x^\sigma}{\partial x'^\rho}\frac{\partial x'^\mu}{\partial x^\kappa}\delta^\kappa_\lambda \frac{\partial}{\partial x^\sigma}\frac{\partial x^\lambda}{\partial x'^\nu} + \frac{\partial x^\sigma}{\partial x'^\rho}\frac{\partial x'^\mu}{\partial x^\kappa}\Gamma^\kappa{}_{\lambda\sigma}\frac{\partial x^\lambda}{\partial x'^\nu} \\ &= \frac{\partial x'^\mu}{\partial x^\kappa}\frac{\partial x^\sigma}{\partial x'^\rho}\frac{\partial}{\partial x^\sigma}\frac{\partial x^\kappa}{\partial x'^\nu} + \frac{\partial x^\sigma}{\partial x'^\rho}\frac{\partial x'^\mu}{\partial x^\kappa}\Gamma^\kappa{}_{\lambda\sigma}\frac{\partial x^\lambda}{\partial x'^\nu} \\ &= \frac{\partial x'^\mu}{\partial x^\kappa}\frac{\partial^2 x^\kappa}{\partial x'^\nu \partial x'^\rho} + \frac{\partial x^\sigma}{\partial x'^\rho}\frac{\partial x'^\mu}{\partial x^\kappa}\Gamma^\kappa{}_{\lambda\sigma}\frac{\partial x^\lambda}{\partial x'^\nu}, \end{aligned}$$

(b) $\omega\omega^{-1} = \mathbb{1}$, $\partial_\mu\omega\omega^{-1} + \omega\partial_\mu\omega^{-1} = 0$, $A'_\mu = -\partial_\mu\omega\omega^{-1} + \omega A_\mu\omega^{-1}$

$$\begin{aligned} \Gamma'^\mu{}_{\nu\rho} &= \frac{\partial x^\sigma}{\partial x'^\rho}\left[-\partial_\sigma\omega^\mu{}_\kappa\delta^\kappa_\lambda(\omega^{-1})^\lambda{}_\nu + \omega^\mu{}_\kappa\Gamma^\kappa{}_{\lambda\sigma}(\omega^{-1})^\lambda{}_\nu\right] \\ &= -\frac{\partial x^\sigma}{\partial x'^\rho}\frac{\partial}{\partial x^\sigma}\frac{\partial x'^\mu}{\partial x^\kappa}\delta^\kappa_\lambda\frac{\partial x^\lambda}{\partial x'^\nu} + \frac{\partial x^\sigma}{\partial x'^\rho}\frac{\partial x'^\mu}{\partial x^\kappa}\Gamma^\kappa{}_{\lambda\sigma}\frac{\partial x^\lambda}{\partial x'^\nu} \\ &= -\frac{\partial x^\kappa}{\partial x'^\nu}\frac{\partial^2 x'^\mu}{\partial x^\kappa\partial x^\sigma}\frac{\partial x^\sigma}{\partial x'^\rho} + \frac{\partial x^\sigma}{\partial x'^\rho}\frac{\partial x'^\mu}{\partial x^\kappa}\Gamma^\kappa{}_{\lambda\sigma}\frac{\partial x^\lambda}{\partial x'^\nu} \\ &= -\frac{\partial^2 x'^\mu}{\partial x'^\nu\partial x'^\rho} + \frac{\partial x^\sigma}{\partial x'^\rho}\frac{\partial x'^\mu}{\partial x^\kappa}\Gamma^\kappa{}_{\lambda\sigma}\frac{\partial x^\lambda}{\partial x'^\nu} \end{aligned}$$

(c) Symmetric part: inhomogeneous transformation rules, not a tensor

(d) Antisymmetric part: homogeneous transformation rules, torsion tensor,

$$S^\rho_{\nu\mu} = \frac{1}{2}(\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu})$$

- **Harmonic gauge:** (harmonic coordinate system)

$$\Gamma^\rho = g^{\mu\nu}\Gamma^\rho{}_{\mu\nu} = 0$$

(a) *Origin:*

$$\Box x^\mu = g^{\rho\nu}D_\rho D_\nu x^\mu = g^{\nu\rho}D_\rho\partial_\nu x^\mu = g^{\nu\rho}(\partial_\rho\partial_\nu x^\mu - \Gamma^\kappa_{\nu\rho}\partial_\kappa x^\mu) = -g^{\nu\rho}\Gamma^\mu_{\nu\rho} = -\Gamma^\mu$$

$\uparrow$

x^μ is a scalar field ($\notin T_p$)

(b) *Gauge transformation:*

$$\begin{aligned}
\Gamma^\mu \to g'^{\nu\rho}\Gamma'^\mu{}_{\nu\rho} &= g^{\tau\sigma}\frac{\partial x'^\nu}{\partial x^\tau}\frac{\partial x'^\rho}{\partial x^\sigma}\left(-\frac{\partial^2 x'^\mu}{\partial x'^\nu \partial x'^\rho} + \frac{\partial x^{\sigma'}}{\partial x'^\rho}\frac{\partial x'^\mu}{\partial x^\kappa}\Gamma^\kappa{}_{\lambda\sigma'}\frac{\partial x^\lambda}{\partial x'^\nu}\right) \\
&= -g^{\tau\sigma}\frac{\partial^2 x'^\mu}{\partial x^\tau \partial x^\sigma} + g^{\tau\sigma}\frac{\partial x'^\mu}{\partial x^\kappa}\Gamma^\kappa{}_{\tau\sigma} \\
&= -g^{\tau\sigma}\frac{\partial^2 x'^\mu}{\partial x^\tau \partial x^\sigma} + \frac{\partial x^\mu}{\partial x'^\kappa}\Gamma^\kappa \\
\implies \quad g^{\tau\sigma}\frac{\partial^2 x'^\mu}{\partial x^\tau \partial x^\sigma} &= \frac{\partial x^\mu}{\partial x'^\kappa}\Gamma^\kappa
\end{aligned}$$

(c) *Always can be reached:* by solving this equation for $x'^\mu(x)$ for given Γ^κ

- **Equivalence Principle:**

(a) *Gauge theory:* at x_0

$$\begin{aligned}
A^\omega_\mu(x) &= \omega(x)(\partial_\mu + A_\mu(x))\omega^{-1}(x), \qquad \omega(x) = e^{(x^\mu - x^\mu_0)A_\mu(x_0)} \\
&= [\mathbb{1} + (x^\mu - x^\mu_0)A_\mu(x_0)](\partial_\mu + A_\mu(x))[\mathbb{1} - (x^\mu - x^\mu_0)A_\mu(x_0)] + \mathcal{O}\left((x - x_0)^2\right) \\
&= \mathcal{O}\left(x - x_0\right)
\end{aligned}$$

(b) *Gravity:* $g_{\mu\nu}(x_0) = \eta_{\mu\nu}$ (global rotations and rescaling $\implies$ Minkowski metric)

new coordinates: $x \to x'$

$$\begin{aligned}
x^\mu - x^\mu_0 &= x'^\mu - x'^\mu_0 - \frac{1}{2}\Gamma^\mu{}_{\nu\rho}(x_0)(x'^\nu - x'^\nu_0)(x'^\rho - x'^\rho_0) \\
\frac{\partial x^\kappa}{\partial x'^\mu} &= \delta^\kappa_\mu - \frac{1}{2}\Gamma^\kappa{}_{\mu\rho}(x_0)(x'^\rho - x'^\rho_0) - \frac{1}{2}\Gamma^\kappa{}_{\nu\mu}(x_0)(x'^\nu - x'^\nu_0) \\
\frac{\partial x^\kappa}{\partial x'^\mu}_{|x'=x'_0} &= \delta^\kappa_\mu, \qquad \frac{\partial^2 x^\kappa}{\partial x'^\nu \partial x'^\rho}_{|x'=x'_0} = -\Gamma^\kappa{}_{\nu\rho}(x_0)
\end{aligned}$$

$$\begin{aligned}
\Gamma'^\mu_{\nu\rho}(x') &= \frac{\partial x'^\mu}{\partial x^\kappa}\frac{\partial^2 x^\kappa}{\partial x'^\nu \partial x'^\rho} + \frac{\partial x^\sigma}{\partial x'^\rho}\frac{\partial x'^\mu}{\partial x^\kappa}\Gamma^\kappa{}_{\lambda\sigma}\frac{\partial x^\lambda}{\partial x'^\nu} \\
&= -\frac{\partial x'^\mu}{\partial x^\kappa}\Gamma^\kappa{}_{\nu\rho}(x_0) + \frac{\partial x^\sigma}{\partial x'^\rho}\frac{\partial x'^\mu}{\partial x^\kappa}\Gamma^\kappa{}_{\lambda\sigma}(x)\frac{\partial x^\lambda}{\partial x'^\nu} \\
\Gamma'^\mu_{\nu\rho}(x'_0) &= -\Gamma^\mu{}_{\nu\rho}(x_0) + \Gamma^\mu{}_{\nu\rho}(x_0) = 0
\end{aligned}$$

2. **Field strength tensor:**

- *Gauge theory:*

$$F_{\mu\nu} = [D_\mu, D_\nu] = [\partial_\mu + \Gamma_\mu, \partial_\nu + \Gamma_\nu] = \partial_\mu\Gamma_\nu - \partial_\nu\Gamma_\mu + [\Gamma_\mu, \Gamma_\nu] = -F_{\nu\mu}$$

- *Curvature tensor:*

$$\boxed{R^\mu{}_{\nu\rho\sigma} = (F_{\rho\sigma})^\mu{}_\nu = \partial_\rho\Gamma^\mu{}_{\nu\sigma} - \partial_\sigma\Gamma^\mu{}_{\nu\rho} + \Gamma^\mu{}_{\kappa\rho}\Gamma^\kappa{}_{\nu\sigma} - \Gamma^\mu{}_{\kappa\sigma}\Gamma^\kappa{}_{\nu\rho}}$$

- *Useful identity for symmetrical connection:*

$$R^{\rho}{}_{\kappa\mu\nu} + R^{\rho}{}_{\mu\nu\kappa} + R^{\rho}{}_{\nu\kappa\mu} = 0.$$

- *Bianchi identity:*
  - Commutators:

$$0 = [A, [B, C]] + [B, [C, A]] + [C, [A, B]],$$

  - Covariant derivative:

$$\begin{aligned} 0 &= [D_\mu, [D_\nu, D_\rho]] + [D_\nu, [D_\rho, D_\mu]] + [D_\rho, [D_\mu, D_\nu]] \\ &= [D_\mu, F_{\nu\rho}] + [D_\nu, F_{\rho\mu}] + [D_\rho, F_{\mu\nu}] \\ &= D_\mu F_{\nu\rho} + D_\nu F_{\rho\mu} + D_\rho F_{\mu\nu}, \end{aligned}$$

  - Curvature:

$$0 = D_\mu R^{\sigma}{}_{\kappa\nu\rho} + D_\nu R^{\sigma}{}_{\kappa\rho\mu} + D_\rho R^{\sigma}{}_{\kappa\mu\nu},$$

- *Ricci tensor:*
  - Gauge theory: internal space external space directions are independent

$$L_A = -\frac{1}{2g^2}\mathrm{tr} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{2g^2} F^{a}{}_{b\mu\nu} F^{b}{}_{a}{}^{\mu\nu}$$

  - Gravity: internal space external space directions are related, simpler contraction possibility

$$\begin{aligned} R^{\mu}{}_{\nu\rho\sigma} &= (F_{\rho\sigma})^{\mu}{}_{\nu} = \partial_\rho \Gamma^{\mu}{}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}{}_{\nu\rho} + \Gamma^{\mu}{}_{\kappa\rho}\Gamma^{\kappa}{}_{\nu\sigma} - \Gamma^{\mu}{}_{\kappa\sigma}\Gamma^{\kappa}{}_{\nu\rho} \\ R_{\nu\sigma} &= R^{\rho}{}_{\nu\rho\sigma} \\ &= \partial_\rho \Gamma^{\rho}{}_{\nu\sigma} - \partial_\sigma \Gamma^{\rho}{}_{\nu\rho} + \Gamma^{\rho}{}_{\kappa\rho}\Gamma^{\kappa}{}_{\nu\sigma} - \Gamma^{\rho}{}_{\kappa\sigma}\Gamma^{\kappa}{}_{\nu\rho} \\ &= -R^{\rho}{}_{\nu\sigma\rho} \end{aligned}$$

  - An alternative:

$$\begin{aligned} R'_{\rho\sigma} &= R^{\mu}{}_{\mu\rho\sigma} \\ &= \partial_\rho \Gamma^{\mu}{}_{\mu\sigma} - \partial_\sigma \Gamma^{\mu}{}_{\mu\rho} + \Gamma^{\mu}{}_{\kappa\rho}\Gamma^{\kappa}{}_{\mu\sigma} - \Gamma^{\mu}{}_{\kappa\sigma}\Gamma^{\kappa}{}_{\mu\rho} \\ &= \partial_\rho \Gamma^{\mu}{}_{\mu\sigma} - \partial_\sigma \Gamma^{\mu}{}_{\mu\rho} \\ &= -R'_{\sigma\rho} \end{aligned}$$

- *Lagrangian for gravity:* $R = g^{\mu\nu} R_{\mu\nu}$, $R' = g^{\mu\nu} R'_{\mu\nu} = 0$ (only if $T = 0$)

E. Metric admissibility

1. **Geometric relation between the affine and the metric structure**

   $T = 0 \implies \{{\rho \atop \mu\nu}\} = \frac{1}{2}(\Gamma^{\rho}_{\nu\mu} + \Gamma^{\rho}_{\mu\nu}) = \Gamma^{\rho}_{\mu\nu}$ (Christoffel symbol)

2. **Parallel transport of two vector fields:** $u^\mu(x)$, $v^\mu(x)$ along $\gamma(s)$

   - *Parallel transport:*

   $$\dot\gamma(s) D_\mu u = \dot\gamma(s) D_\mu v = 0.$$

   - *Scalar product preserved:*

   $$\dot\gamma(s) D_\mu u^\nu g_{\nu\rho} v^\rho = u^\nu v^\rho \dot\gamma(s) D_\mu g_{\nu\rho} = 0$$

3. **Metric admissibility:**

   $$Dg = 0$$

4. **Solution for the Chistoffel symbols:** $\Gamma_{\rho\mu\nu} = g_{\rho\kappa}\Gamma^{\kappa}_{\mu\nu}$

$$\begin{aligned}
D_\rho g_{\mu\nu} &= \partial_\rho g_{\mu\nu} - g_{\kappa\nu}\Gamma^{\kappa}_{\mu\rho} - g_{\mu\kappa}\Gamma^{\kappa}_{\nu\rho} \\
D_\mu g_{\nu\rho} &= \partial_\mu g_{\nu\rho} - g_{\kappa\rho}\Gamma^{\kappa}_{\nu\mu} - g_{\nu\kappa}\Gamma^{\kappa}_{\rho\mu} \\
D_\nu g_{\rho\mu} &= \partial_\nu g_{\rho\mu} - g_{\kappa\mu}\Gamma^{\kappa}_{\rho\nu} - g_{\rho\kappa}\Gamma^{\kappa}_{\mu\nu} \\
0 = D_\mu g_{\nu\rho} + D_\nu g_{\rho\mu} - D_\rho g_{\mu\nu} &= \partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \underbrace{-g_{\kappa\rho}\Gamma^{\kappa}_{\nu\mu} - g_{\rho\kappa}\Gamma^{\kappa}_{\mu\nu}}_{-\Gamma_{\rho\mu\nu} - \Gamma_{\rho\nu\mu}} \\
\Gamma_{\rho\mu\nu} + \Gamma_{\rho\nu\mu} &= \partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}
\end{aligned}$$

$$\boxed{\{{\rho \atop \mu\nu}\} = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})}$$

5. **Example:** S_2, $x^\mu = (\theta, \phi)$

   - *Invariant length:*

   $$ds^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

   - *Metric tensor:*

   $$g_{\mu\nu} = r^2 \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}, \quad g^{\mu\nu} = \frac{1}{r^2} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sin^2\theta} \end{pmatrix}.$$

- *Christoffel symbol:*

$$\left\{ {\theta \atop \phi\phi} \right\} = -\sin\theta\cos\theta, \quad \left\{ {\phi \atop \theta\phi} \right\} = \left\{ {\phi \atop \phi\theta} \right\} = -\cot\theta$$

- *Curvature:*

$$\begin{aligned} R^{\theta}_{\phi\theta\phi} &= \partial_\theta \left\{ {\theta \atop \phi\phi} \right\} - \partial_\phi \left\{ {\theta \atop \theta\phi} \right\} + \left\{ {\theta \atop \theta\mu} \right\} \left\{ {\mu \atop \phi\phi} \right\} - \left\{ {\theta \atop \phi\mu} \right\} \left\{ {\mu \atop \theta\phi} \right\} \\ &= -\partial_\theta \sin\theta\cos\theta + \sin\theta\cos\theta\cot\theta = \sin^2\theta \\ R^{\phi}_{\theta\theta\phi} &= g^{\phi\phi} R_{\phi\theta\theta\phi} = -g^{\phi\phi} R_{\theta\phi\theta\phi} = -g^{\phi\phi} g_{\theta\theta} R^{\theta}_{\phi\theta\phi} = -1 \end{aligned}$$

- *Ricci tensor:*

$$R = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

- *Scalar curvature:* $R = \frac{2}{r^2}$

6. **Metric admissible curvature tensor:**

$$R_{\rho\kappa\mu\nu} = -R_{\kappa\rho\mu\nu} = R_{\mu\nu\rho\kappa}$$

and the number of independent components is $256 \to 20$. We note that the curvature is vanishing for flat space only.

7. **Double contracted Bianchy identity:**

$$\begin{aligned} 0 &= D_\mu R^{\sigma}{}_{\kappa\nu\rho} + D_\nu R^{\sigma}{}_{\kappa\rho\mu} + D_\rho R^{\sigma}{}_{\kappa\mu\nu} \qquad \sigma \leftrightarrow \mu \\ 0 &= D_\mu R_{\kappa\rho} + D_\nu R^{\nu}{}_{\kappa\rho\mu} - D_\rho R_{\kappa\mu} \qquad \kappa \leftrightarrow \mu \\ 0 &= 2D_\mu R^{\mu}{}_{\rho} - D_\rho R \end{aligned}$$

8. **Einstein tensor:**

$$\begin{aligned} G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \\ D_\mu G^{\mu}{}_{\nu} &= 0 \end{aligned}$$

F. Technical details

1. **Gauge invariance of the action:**

$$\begin{aligned} S &= \int d^4x L \\ d^4x &\to d^4x' \left| \det \frac{\partial x}{\partial x'} \right| \neq d^4x' \end{aligned}$$

2. **Invariant integral:** $g = \det g_{\mu\nu}$

$$\begin{aligned} g_{\mu\nu} &= \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} g'_{\rho\sigma} \\ g &= g' \left(\det \frac{\partial x'}{\partial x} \right)^2 \\ d_{\text{inv}} x &= dx\sqrt{-g} \to dx' \left| \det \frac{\partial x}{\partial x'} \right| \sqrt{-g'} \left| \det \frac{\partial x'}{\partial x} \right| = dx' \sqrt{-g'} \end{aligned}$$

3. **Minor matrix:**

$$(M_A)_{j,k} = (-1)^{j+k} d_{k,j},$$

$$\nearrow$$

determinant of the matrix, obtained by omitting the j-th row and the k-th column of A.

- *Determinant of A:*

$$\begin{aligned} \det[A] &= \sum_j A_{k,j} (-1)^{j+k} d_{k,j} \quad \leftarrow \quad \text{expansion along a row} \\ &= \sum_k A_{k,j} (-1)^{j+k} d_{k,j} \quad \leftarrow \quad \text{expansion along a culomn} \end{aligned}$$

- *Two identical rows:* $A_{\ell,j} = A_{k,j}$

$$\sum_j A_{k,j} (-1)^{j+k} d_{\ell,j} = \det[A] = 0$$

- *Summary:*

$$A^{-1} = \frac{M_A}{\det[A]}$$

4. **Variation of the metric tensor:**

$$\begin{aligned} \frac{\partial \det[A]}{\partial A_{k,j}} &= (-1)^{j+k} d_{k,j} = (M_A)_{j,k} \\ \delta g &= \frac{\partial g}{\partial g_{\sigma\mu}} \delta g_{\sigma\mu} = \underbrace{g g^{\sigma\mu}}_{M_g} \delta g_{\mu\sigma} \\ \partial_\nu g &= g g^{\sigma\mu} \partial_\nu g_{\sigma\mu} \end{aligned}$$

5. **Divergence of a vector field:** (no torsion)

$$\begin{aligned} \Gamma^{\mu}_{\nu\mu} &= \frac{1}{2} g^{\sigma\mu} (\partial_\nu g_{\sigma\mu} + \partial_\mu g_{\nu\sigma} - \partial_\sigma g_{\nu\mu}) = \frac{1}{2} g^{\sigma\mu} \partial_\nu g_{\sigma\mu} = \frac{\partial_\nu g}{2g} = \frac{\partial_\nu \sqrt{-g}}{\sqrt{-g}} \\ D_\mu v^\mu &= \partial_\mu v^\mu + \Gamma^{\mu}_{\nu\mu} v^\nu = \partial_\mu v^\mu + \frac{\partial_\mu \sqrt{-g}}{\sqrt{-g}} v^\mu = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} v^\mu) \\ \int dx \sqrt{-g} D_\mu v^\mu &= \int dx \partial_\mu (\sqrt{-g} v^\mu) = \int ds_\mu \sqrt{-g} v^\mu \\ D_\mu j^\mu &= 0 \quad \Longrightarrow \quad \partial_\mu (\sqrt{-g} j^\mu) = 0, \qquad Q = \int d^3x \sqrt{-g} j^0 \end{aligned}$$

6. **Metric admissibility:**

$$D_\mu D^\mu = g^{\mu\nu} D_\mu D_\nu = D_\mu g^{\mu\nu} D_\nu = D_\mu D_\nu g^{\mu\nu}$$
$$D_\mu D^\mu \phi = D_\mu \partial^\mu \phi = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} \partial^\mu \phi)$$

G. Dynamics

1. **Einstein-Hilbert action:**

$$\boxed{S_E = -\frac{1}{16\pi G} \int dx \sqrt{-g}(R + 2\Lambda) = -\frac{1}{16\pi G} \int dx \sqrt{-g}(g^{\mu\nu} R_{\mu\nu} + 2\Lambda)}$$

2. **Independent fields:**

(a) *Affine connection:*

tensor

↘

$$\Gamma^\rho_{\mu\nu} = \left\{ {\rho \atop \mu\nu} \right\} + C^\rho_{\mu\nu}$$
$$\tilde{D}_\mu v^\nu = \partial_\mu v^\nu + \left\{ {\nu \atop \rho\mu} \right\} v^\rho$$

(b) *Metric tensor:* $g^{\mu\nu}$

$$\delta^\mu_\rho = g^{\mu\nu} g_{\nu\rho}$$
$$0 = \delta g^{\mu\nu} g_{\nu\rho} + g^{\mu\nu} \delta g_{\nu\rho}$$
$$\delta g_{\mu\nu} = -g_{\mu\rho} g_{\nu\sigma} \delta g^{\rho\sigma}$$

thus the metric can not be used in this calculation to change the position of the indices. Instead, we keep as the independent tensor field for the metric.

3. **Variation of the field strength tensor:** $C \to C + \delta C$

(a) *Gauge theory:*

$$\begin{aligned}
F_{\mu\nu} &= [D_\mu, D_\nu] = [\tilde{D}_\mu + C_\mu, \tilde{D} + C_\nu] \\
\delta F_{\mu\nu} &= [D_\mu + \delta C_\mu, D_\nu + \delta C_\nu] - [D_\mu, D_\nu] \\
&= (D_\mu + \delta C_\mu)(D_\nu + \delta C_\nu) - (D_\nu + \delta C_\nu)(D_\mu + \delta C_\mu) - D_\mu D_\nu + D_\nu D_\mu \\
&= D_\mu \delta C_\nu + \delta C_\mu D_\nu - D_\nu \delta C_\mu - \delta C_\nu D_\mu + \mathcal{O}\left(\delta C^2\right) \\
&= (D_\mu \delta C_\nu) - (D_\nu \delta C_\mu) + \mathcal{O}\left(\delta C^2\right)
\end{aligned}$$

(b) *Gravity:*

$$\delta R^\mu{}_{\nu\rho\sigma} = D_\rho \delta C^\mu{}_{\nu\sigma} - D_\sigma \delta C^\mu{}_{\nu\rho}$$
$$\delta R_{\nu\sigma} = D_\rho \delta C^\rho{}_{\nu\sigma} - D_\sigma \delta C^\rho{}_{\nu\rho}$$

(c) *Covariance:*

$$g^{\nu\sigma}\delta R_{\nu\sigma} = K\delta C + \partial\delta C = K_\kappa{}^{\nu\rho}\delta C^\kappa{}_{\nu\rho} + \tilde{D}_\rho v^\rho$$

↗ ↑ ↖

scalar tensor vector

(d) *Non-singular K:* $K(C)$ is linear in C and $K = 0 \implies C = 0$

4. **Variation of the integrand:**

$$\delta\sqrt{-g} = -\frac{g}{2\sqrt{-g}} g^{\mu\nu}\delta g_{\mu\nu} = -\frac{1}{2}\sqrt{-g} g^{\mu\nu} g_{\mu\rho} g_{\nu\sigma}\delta g^{\rho\sigma} = -\frac{1}{2}\sqrt{-g} g_{\nu\sigma}\delta g^{\nu\sigma}$$
$$\delta[\sqrt{-g}(R_{\mu\nu}g^{\mu\nu} + 2\Lambda)] = \delta\sqrt{-g}(R_{\mu\nu}g^{\mu\nu} + 2\Lambda) + \sqrt{-g}\delta R_{\mu\nu}g^{\mu\nu} + \sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu}$$
$$= -\frac{1}{2}\sqrt{-g} g_{\nu\sigma}\delta g^{\nu\sigma}(R + 2\Lambda) + \sqrt{-g}\tilde{D}_\rho v^\rho + \sqrt{-g}\delta C^\kappa{}_{\nu\rho} K_\kappa{}^{\nu\rho} + \sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu}$$
$$= \sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \Lambda g_{\mu\nu}\right)\delta g^{\mu\nu} + \sqrt{-g}\delta C^\kappa{}_{\nu\rho} K_\kappa{}^{\nu\rho} + \sqrt{-g}\tilde{D}_\rho v^\rho$$

↑ ↗ ↑

Einstein equation metric admissibility surfacce term, ignored

↘

$$\boxed{G_{\mu\nu} - \Lambda g_{\mu\nu} = 0}$$

V. COUPLING TO MATTER

A. Point particle in an external gravitational field

1. **Equivalence Principle:** no gravitational field at a selected space-time point

$$\frac{d\dot{\xi}^a(s)}{ds} = 0$$

- *Filling up the space-time with the solutions:* $u^\mu(x) = \dot{x}^\mu$

$$\boxed{u^\nu D_\nu u^\mu = \dot{u}^\mu + u^\rho \Gamma^\mu{}_{\rho\nu} u^\nu = 0}$$

- *Non-gravitational force:*

$$\begin{aligned}\dot{u}^\mu + u^\rho \Gamma^\mu{}_{\rho\nu} u^\nu &= \frac{F^\mu}{mc} \\ mc\dot{u}^\mu &= F^\mu + F^\mu_{gr} \\ F^\mu_{gr} &= -mu^\mu \Gamma^\mu{}_{\rho\nu} u^\nu\end{aligned}$$

Lorentz force of electrodynamics:

$$\begin{aligned}mc\ddot{x}^\mu &= F^\mu + F^\mu_{ed} \\ F^\mu_{ed} &= \frac{e}{c} F^\mu{}_\nu u^\nu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\end{aligned}$$

2. **Spin precession:**

- *Rest frame:* $S^\mu = (0, \mathbf{S})$, $S_\mu \dot{x}^\mu = 0$
- *E.O.M.:*

$$0 = \frac{dS^\mu}{ds} \rightarrow \dot{S}^\mu + \Gamma^\mu{}_{\rho\nu} S^\rho \dot{x}^\nu$$

- *Auxiliary condition:*

$$S_\mu \dot{x}^\mu = 0$$

- *External force without torque:*

$$\begin{aligned}\frac{d\mathbf{S}}{dt} &= 0 \quad \leftarrow \quad \mathrm{co-moving\ frame:} \quad \dot{S} = a\dot{x} \\ 0 &= a\dot{x}_\mu \dot{x}^\mu + S_\mu \ddot{x}^\mu \quad \Longrightarrow \quad a = -S_\mu \ddot{x}^\mu = -S_\mu \frac{F^\mu}{mc}\end{aligned}$$

- *Fermi-Walker transport:*

$$\boxed{\dot{S}^\mu = -S_\nu \frac{F^\nu}{mc} \dot{x}^\mu}$$

3. **Variational equation of motion:**

- *Action:* $\dot{x}^\mu(\tau) = \frac{dx^\mu(\tau)}{d\tau}$ ($\tau \neq s$ to leave the variation of all components of x^μ independent)

$$S = -mc \int \sqrt{\dot{x}^\mu g_{\mu\nu}(x) \dot{x}^\nu} d\tau$$

- *Euler-Lagrange equation:*

$$
\begin{aligned}
\frac{\partial L}{\partial x^\rho} &= -mc\frac{\dot{x}^\mu \partial_\rho g_{\mu\nu}\dot{x}^\nu}{2\sqrt{\dot{x}^\mu g_{\mu\nu}\dot{x}^\nu}}, \\
\frac{\partial L}{\partial \frac{dx^\rho}{d\tau}} &= -mc\frac{g_{\rho\nu}\dot{x}^\nu}{\sqrt{\dot{x}^\mu g_{\mu\nu}\dot{x}^\nu}} \\
0 &= \frac{\partial L}{\partial x^\rho} - \frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}^\rho} \\
&= -\frac{\dot{x}^\mu \partial_\rho g_{\mu\nu}\dot{x}^\nu}{2\sqrt{\dot{x}^\mu g_{\mu\nu}\dot{x}^\nu}} + \frac{d}{d\tau}\frac{g_{\rho\nu}\dot{x}^\nu}{\sqrt{\dot{x}^\mu g_{\mu\nu}\dot{x}^\nu}} \\
&= \frac{1}{\sqrt{\dot{x}^\mu g_{\mu\nu}\dot{x}^\nu}}\left[-\frac{1}{2}\dot{x}^\mu \partial_\rho g_{\mu\nu}\dot{x}^\nu + \dot{x}^\kappa \partial_\kappa g_{\rho\nu}\dot{x}^\nu + g_{\rho\nu}\ddot{x}^\nu + g_{\rho\nu}\dot{x}^\nu\frac{d}{d\tau}\frac{1}{\sqrt{\dot{x}^\mu g_{\mu\lambda}\dot{x}^\lambda}}\right]
\end{aligned}
$$

$\uparrow$

symmetrize in κ and ν the factor $\partial_\kappa g_{\rho\nu}$

$$
\begin{aligned}
0 &= -\frac{1}{2}\dot{x}^\mu \partial_\rho g_{\mu\nu}\dot{x}^\nu + \frac{1}{2}\dot{x}^\kappa(\partial_\kappa g_{\rho\nu} + \partial_\nu g_{\rho\kappa})\dot{x}^\nu + g_{\rho\nu}\ddot{x}^\nu + g_{\rho\nu}\frac{dx^\nu}{d\tau}\frac{d}{d\tau}\frac{1}{\sqrt{\dot{x}^\mu g_{\mu\lambda}\dot{x}^\lambda}} \\
&= g_{\rho\sigma}(\ddot{x}^\sigma + \Gamma^\sigma_{\nu\kappa}\dot{x}^\nu\dot{x}^\kappa + \dot{x}^\sigma\dot{f}), \quad f(\tau) = \frac{1}{\sqrt{\dot{x}^\mu g_{\mu\lambda}\dot{x}^\lambda}}
\end{aligned}
$$

$\nearrow$

Lagrange multiplier for the constraint $\dot{x}^2 = 1$

- *Geodesics, $\tau = s$:*

$$u^\nu D_\nu u^\mu = \dot{u}^\mu + u^\rho \Gamma^\mu{}_{\rho\nu} u^\nu = 0$$

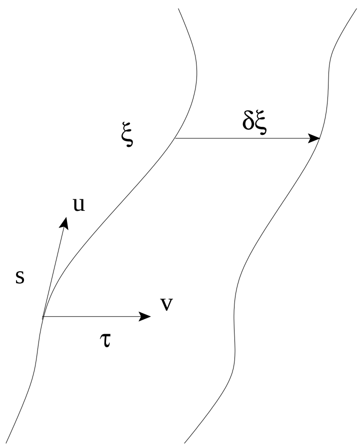
4. **Geodesic deviation:**

- *Newton equation:*

$$
\begin{aligned}
m\ddot{\xi}^j &= -\nabla^j U \\
\xi(t) &\to \xi(t) + \delta\xi(t) \\
m\delta\ddot{\xi}^j &= -\delta\xi^k \nabla^k \nabla^j U
\end{aligned}
$$

- *Geodesics:*
  - Fill up the space-time with geodesics $\xi(s)$, $u(x) = \frac{d\xi(x)}{ds} = \dot{\xi}(x)$, $\frac{d\phi}{ds} = u^\nu D_\nu \phi = \dot{\phi}$
  - The space-time strip, bounded by $\xi(s)$ and $\xi(s) + \delta\xi(s)$: $\xi^\mu(s,\tau)$
  - Holonomic coordinate axis vectors: $u = \partial_s\xi(s,\tau)$, $v = \partial_\tau\xi(s,\tau)$

$$
\begin{aligned}
\partial_s v &= \partial_\tau u \\
u^\nu D_\nu v^\mu &= v^\nu D_\nu u^\mu
\end{aligned}
$$



– Deviation equation for $\delta\xi = \epsilon\partial_\tau\xi = \epsilon v$:

$$\begin{aligned}
\ddot{v} &= u^\mu D_\mu(u^\nu D_\nu v) \\
&= u^\mu D_\mu(v^\nu D_\nu u) \quad \leftarrow \quad \text{holonomy} \\
&= u^\mu(D_\mu v^\nu)D_\nu u + u^\mu v^\nu D_\mu D_\nu u \quad \leftarrow \quad \text{Leibnitz rule} \\
&= v^\mu(D_\mu u^\nu)D_\nu u + u^\mu v^\nu [D_\mu, D_\nu]u + u^\mu v^\nu D_\nu D_\mu u \quad \leftarrow \quad \text{holonomy + alg. id.} \\
&= v^\mu(D_\mu u^\nu)D_\nu u + u^\mu v^\nu [D_\mu, D_\nu]u + v^\nu D_\nu(u^\mu D_\mu u) - v^\nu(D_\nu u^\mu)D_\mu u \quad \leftarrow \quad \text{Leibnitz rule} \\
&= u^\mu v^\nu [D_\mu, D_\nu]u + v^\nu D_\nu(u^\mu D_\mu u) \quad \leftarrow \quad \text{alg. cancellation} \\
\ddot{v}^\rho &= R^\rho{}_{\kappa\mu\nu} u^\kappa u^\mu v^\nu \quad \leftarrow \quad u^\mu D_\mu u = \ddot{\xi} = 0
\end{aligned}$$

- *Electrodynamics:*

$$\begin{aligned}
mc\ddot{x}_n^\mu(s) &= \frac{e}{c} F_{\mu\nu}\dot{x}_n^\nu(s) \\
mc(\ddot{x}^\mu + \delta\ddot{x}^\mu) &= \frac{e}{c} F^\mu{}_\nu(x + \delta x)(\dot{x}^\nu + \delta\dot{x}^\nu) \\
mc\delta\ddot{x}^\mu &= \frac{e}{c}\delta x^\rho \partial_\rho F^\mu{}_\nu \dot{x}^\nu + \frac{e}{c} F^\mu{}_\nu \delta\dot{x}^\nu
\end{aligned}$$

5. **Newtonian limit:** slow test particle, $\frac{dx^\mu}{ds} \approx (1, 0, 0, 0)$, and weak, static gravitational field

$$\begin{aligned}
g_{\mu\nu} &= \eta_{\mu\nu} + \gamma_{\mu\nu}, \quad g^{\mu\nu} = \eta^{\mu\nu} - \gamma^{\mu\nu} \\
\ddot{x}^\mu \approx -\Gamma^\mu_{00} &= \frac{1}{2}\frac{\partial\gamma_{00}}{\partial x_\mu} \quad \leftarrow \quad \Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) \\
\ddot{\mathbf{x}} &= -\nabla\phi
\end{aligned}$$

$$\boxed{\phi = \frac{\gamma_{00}}{2}}$$