Elements of Quantum Field Theory

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(Dated: October 13, 2020)

Video: `https://drive.google.com/drive/folders/1_kKM4lV1ueGBBmd59TjgXhEgHwcD2FWo?usp=sharing`

Contents

Goal of the course: Bird's eye view of quantum field theory, until Feynman's rule

Why classical field theory:

No action at a distance in special relativity $\implies$ retardation $\implies$ dynamical degrees of freedom in space

Why quantum field theory:

1. Number of particles is a dynamical degree of freedom:
   - relativistic pair creation or annihilation
   - excitations of ion lattice

   $\implies$ Schrödinger's equation, $i\hbar\partial_t\psi(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N) = H\psi(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N)$, is not sufficient
2. The particle number is ill defined in bound states
   - non-relativistic physics:
     - bound states need infinitely high order of the perturbation expansion in the attractive potential
     - each order represents an exchange of particle
   - relativistic physics: retarded interaction = particle exchange
3. A 'technical' problem: Difficult to handle the (anti)symmetrised states of equivalent particles ($n!$ increases too fast)

$$\begin{aligned}\psi_\xi(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N) &= \sum_{\pi\in S_N}\xi^{\sigma(\pi)}\psi_1(\boldsymbol{x}_{\pi(1)})\cdots\psi_N(\boldsymbol{x}_{\pi(N)}) \quad \leftarrow \quad N! \text{ terms}\\ \langle\psi_\xi|A|\phi_\xi\rangle &= \cdots \quad \leftarrow \quad N!^2 \text{ terms}\end{aligned}$$

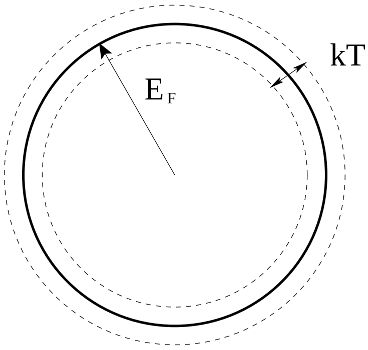
4. Quantum Field Theory provides a (unique) representation of the space-time symmetries in terms of particles

Why relativistic quantum field theory:

1. High energy physics: relativistic kinematics
2. Condensed matter physics: linear dispersion relation, $E(\boldsymbol{p}) = c\sqrt{m^2c^2+\boldsymbol{p}^2} \to c|\boldsymbol{p}|$ as $|\boldsymbol{p}|\to\infty$ or $m\to 0$
   (a) Acoustic phonons, $E(\boldsymbol{p}) = c|\boldsymbol{p}|$
   (b) Fermonic systems $E(\boldsymbol{p}) = \frac{\boldsymbol{p}^2}{2m} - \mu = (\boldsymbol{p}-\boldsymbol{p}_F)\frac{\boldsymbol{p}_F}{m} + \cdots - \mu$

Relativistic quantum field theory:

1. Particles and anti-particles / holes



2. Interaction mixes particles and anti-particles / holes

3. Impossible to fix the number of degrees of freedom $\implies$ no RQM

4. Fixed number of degrees of freedom in free dynamics

   - Conserved momentum can be measured without demolishing the state
   - Coordinate measurement changes the state
   - Momentum representation: $\hat{\boldsymbol{p}} = \boldsymbol{p}$, $\hat{\boldsymbol{x}} = i\hbar\frac{\partial}{\partial \boldsymbol{p}} + \boldsymbol{c} \neq \hat{\boldsymbol{x}}_{Sch}$

     $\uparrow$

     particles $\longleftrightarrow$ anti-particles / holes
   - Quantum field $\phi(t, \boldsymbol{x})$:
     - $\boldsymbol{x}$ is only a formal variable
     - $\boldsymbol{p}$ has physical interpretation

I. BASIC IDEA OF QUANTUM FIELD THEORY

A. Quantum mechanical representation of a set of N free particles

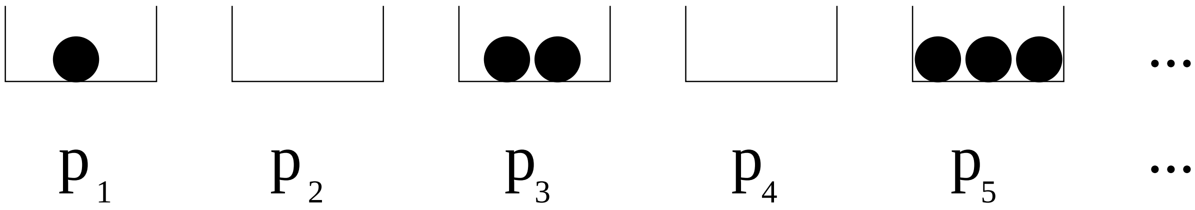
1. **Occupation number representation:** $n(\boldsymbol{p})$ particles with momentum $\boldsymbol{p}$

2. **Energy-momentum spectrum of free particles:** ($\hbar = c = 1$)

$$E(\boldsymbol{p}) = \begin{cases} \frac{\boldsymbol{p}^2}{2m} & \text{non}-\text{relativistic} \\ \sqrt{m^2 + \boldsymbol{p}^2} & \text{relativistic} \end{cases}$$

$$E = \sum_{j=1}^{N} E(\boldsymbol{p}_j) = \sum_{\boldsymbol{p}} n(\boldsymbol{p}) E(\boldsymbol{p}), \qquad \boldsymbol{P} = \sum_{j=1}^{N} \boldsymbol{p}_j = \sum_{\boldsymbol{p}} n(\boldsymbol{p}) \boldsymbol{p}$$

3. **Harmonic oscillator for each momentum:** equidistant spectrum $\implies$ harmonic oscillator



- *Example:* $n(\boldsymbol{p}_1) = 1$, $n(\boldsymbol{p}_2) = 0$, $n(\boldsymbol{p}_3) = 2$, $n(\boldsymbol{p}_4) = 0$, $n(\boldsymbol{p}_5) = 3$, ...,
- *Occupation number representation:* $|\boldsymbol{p}_1, \boldsymbol{p}_3, \boldsymbol{p}_3, \boldsymbol{p}_5, \boldsymbol{p}_5, \boldsymbol{p}_5\rangle \quad \Longrightarrow \quad |n(\boldsymbol{p}\rangle$
- *Symmetrization of equivalent particles:* trivial
- *Canonical variables:* $X_{\boldsymbol{p}}$ and $P_{\boldsymbol{q}}$,

$$[X(\boldsymbol{p}), P(\boldsymbol{q})]_\xi = i\delta(\boldsymbol{p} - \boldsymbol{q}), \qquad [X(\boldsymbol{p}), X(\boldsymbol{q})]_\xi = [P(\boldsymbol{p}), P(\boldsymbol{q})]_\xi = 0$$

  where

$$[A, B]_\xi = AB - \xi BA,$$

  with $\xi = 1$ (bosons) $\xi = -1$ (fermions)

- *Hamiltonian:*

$$H = \int \frac{d^3p}{(2\pi)^3} \left(\frac{P^2(\boldsymbol{p})}{2M(\boldsymbol{p})} + \frac{M(\boldsymbol{p})\omega^2(\boldsymbol{p})}{2} X^2(\boldsymbol{p}) \right)$$

  Spectrum:

$$\begin{aligned} E &= \int \frac{d^3p}{(2\pi)^3} \omega(\boldsymbol{p}) \left[n(\boldsymbol{p}) + \frac{1}{2} \right] \\ \omega(\boldsymbol{p}) &= E(\boldsymbol{p}) = \sqrt{m^2 + \boldsymbol{p}^2}, \end{aligned}$$

  with arbitrary $M(\boldsymbol{p})$.

4. **Particle creation and destruction operators:**

$$\begin{aligned} a(\boldsymbol{p}) &= \sqrt{(2\pi)^3 2\omega(\boldsymbol{p})} \frac{M(\boldsymbol{p})\omega(\boldsymbol{p})X(\boldsymbol{p}) + iP(\boldsymbol{p})}{\sqrt{2M(\boldsymbol{p})\omega(\boldsymbol{p})}} \\ a^\dagger(\boldsymbol{p}) &= \sqrt{(2\pi)^3 2\omega(\boldsymbol{p})} \frac{M(\boldsymbol{p})\omega(\boldsymbol{p})X(\boldsymbol{p}) - iP(\boldsymbol{p})}{\sqrt{2M(\boldsymbol{p})\omega(\boldsymbol{p})}} \end{aligned}$$

$$\begin{aligned} \left[a(\boldsymbol{p}), a^\dagger(\boldsymbol{q})\right]_\xi &= (2\pi)^3 2\sqrt{\omega(\boldsymbol{p})\omega(\boldsymbol{q})} \frac{M(\boldsymbol{p})\omega(\boldsymbol{p})(-i)[X(\boldsymbol{p}), P(\boldsymbol{q})]_\xi + M(\boldsymbol{q})\omega(\boldsymbol{q})i[P(\boldsymbol{p}), X(\boldsymbol{q})]_\xi}{2\sqrt{M(\boldsymbol{p})\omega(\boldsymbol{p})M(\boldsymbol{q})\omega(\boldsymbol{q})}} \\ &= (2\pi)^3 2\omega(\boldsymbol{p})\delta(\boldsymbol{p} - \boldsymbol{q}) \\ [a(\boldsymbol{p}), a(\boldsymbol{q})]_\xi &= \left[a^\dagger(\boldsymbol{p}), a^\dagger(\boldsymbol{q})\right]_\xi = 0, \end{aligned}$$

5. **Fock space:** $\forall \boldsymbol{p}\ \mathcal{H}_{\boldsymbol{p}}$ is the Hilbert space of the harmonic oscillator of $\boldsymbol{p}$

$$\mathcal{H} = \mathcal{H}_{\boldsymbol{p}_1} \otimes \mathcal{H}_{\boldsymbol{p}_2} \otimes \cdots = \otimes \prod_{\boldsymbol{p}} \mathcal{H}_{\boldsymbol{p}}$$

- *Bosons:*
  - No particle state: vacuum, ground state

$$|0\rangle = |0\rangle_{\boldsymbol{p}_1} \otimes |0\rangle_{\boldsymbol{p}_2} \otimes \cdots = \otimes \prod_{\boldsymbol{p}} |0\rangle_{\boldsymbol{p}}$$

  - One particle:

$$\begin{aligned} |\boldsymbol{p}\rangle &= a^\dagger(\boldsymbol{p})|0\rangle \\ |\Psi_1\rangle &= \int \underbrace{\frac{d\boldsymbol{p}}{(2\pi)^3 2\omega_{\boldsymbol{p}}}}_{\tilde{d\boldsymbol{p}}} \Psi_1(\boldsymbol{p})|\boldsymbol{p}\rangle = \int \tilde{d\boldsymbol{p}} \Psi_1(\boldsymbol{p})|\boldsymbol{p}\rangle = \int \tilde{d\boldsymbol{p}} \Psi_1(\boldsymbol{p}) a^\dagger(\boldsymbol{p})|0\rangle \end{aligned}$$

↑

wave function in momentum space

  - A two-particles:

$$\begin{aligned} |\boldsymbol{p}_1, \boldsymbol{p}_2\rangle &= a^\dagger(\boldsymbol{p}_1) a^\dagger(\boldsymbol{p}_2)|0\rangle \\ |\Psi_2\rangle &= \frac{1}{2}\int \tilde{d\boldsymbol{p}_1}\tilde{d\boldsymbol{p}_2} \Psi_2(\boldsymbol{p}_1, \boldsymbol{p}_2)|\boldsymbol{p}_1, \boldsymbol{p}_2\rangle = \frac{1}{2}\int \tilde{d\boldsymbol{p}_1}\tilde{d\boldsymbol{p}_2} \Psi_2(\boldsymbol{p}_1, \boldsymbol{p}_2) a^\dagger(\boldsymbol{p}_1) a^\dagger(\boldsymbol{p}_2)|0\rangle \end{aligned}$$

↑

wave function in momentum space

Exchange symmetry:

$$\begin{aligned} |\boldsymbol{p}_1, \boldsymbol{p}_2\rangle &= a^\dagger(\boldsymbol{p}_1) a^\dagger(\boldsymbol{p}_2)|0\rangle = |\boldsymbol{p}_2, \boldsymbol{p}_1\rangle = a^\dagger(\boldsymbol{p}_2) a^\dagger(\boldsymbol{p}_1)|0\rangle \\ 0 &= (a^\dagger(\boldsymbol{p}_1) a^\dagger(\boldsymbol{p}_2) - a^\dagger(\boldsymbol{p}_2) a^\dagger(\boldsymbol{p}_1))|0\rangle = [a^\dagger(\boldsymbol{p}_2), a^\dagger(\boldsymbol{p}_1)]_+|0\rangle \quad \to \quad \xi = +1 \end{aligned}$$

  - Arbitrary number of particles: Bound state, condensate, SSB, Higgs mechanism,...

$$|\Psi\rangle = \left[\Psi_0 + \int \tilde{d\boldsymbol{p}} \Psi_1(\boldsymbol{p}) a^\dagger(\boldsymbol{p}) + \frac{1}{2}\int \tilde{d\boldsymbol{p}_1}\tilde{d\boldsymbol{p}_2} \Psi_2(\boldsymbol{p}_1, \boldsymbol{p}_2) a^\dagger(\boldsymbol{p}_1) a^\dagger(\boldsymbol{p}_2) + \cdots\right]|0\rangle.$$

- *Fermions:*

$$\begin{aligned} |\Psi_1\rangle &= \int \frac{d^3p}{(2\pi)^3}\frac{m}{\omega_{\boldsymbol{p}}} \Psi^p_{1\alpha}(\boldsymbol{p}) c^\dagger_\alpha(\boldsymbol{p})|0\rangle \\ |\Psi_2\rangle &= \frac{1}{2}\int \frac{d^3p_1}{(2\pi)^3}\frac{m}{\omega_{\boldsymbol{p}_1}}\frac{d^3p_1}{(2\pi)^3}\frac{m}{\omega_{\boldsymbol{p}_1}} \Psi_{2\alpha\beta}(\boldsymbol{p}_1, \boldsymbol{p}_2) c^\dagger_\alpha(\boldsymbol{p}_1) c^\dagger_\beta(\boldsymbol{p}_2)|0\rangle \end{aligned}$$

Exchange symmetry:

$$\begin{aligned} |\boldsymbol{p}_1, \boldsymbol{p}_2\rangle &= c^\dagger(\boldsymbol{p}_1) c^\dagger(\boldsymbol{p}_2)|0\rangle = -|\boldsymbol{p}_2, \boldsymbol{p}_1\rangle = -c^\dagger(\boldsymbol{p}_2) c^\dagger(\boldsymbol{p}_1)|0\rangle \\ 0 &= (c^\dagger(\boldsymbol{p}_1) c^\dagger(\boldsymbol{p}_2) + c^\dagger(\boldsymbol{p}_2) c^\dagger(\boldsymbol{p}_1))|0\rangle = [c^\dagger(\boldsymbol{p}_2), c^\dagger(\boldsymbol{p}_1)]_-|0\rangle \quad \to \quad \xi = -1 \end{aligned}$$

6. **Particle number operator:**

$$\begin{aligned}
N &= \int \tilde{d\boldsymbol{k}} a^\dagger(\boldsymbol{k}) a(\boldsymbol{k}) \\
N|\Psi_n\rangle &= n|\Psi_n\rangle \\
N|0\rangle &= \int \tilde{d\boldsymbol{k}} a^\dagger(\boldsymbol{k}) a(\boldsymbol{k})|0\rangle = 0 = 0|0\rangle \\
N|\Psi_1\rangle &= \int \tilde{d\boldsymbol{k}} a^\dagger(\boldsymbol{k}) a(\boldsymbol{k}) \int \tilde{d\boldsymbol{p}} \Psi_1(\boldsymbol{p}) a^\dagger(\boldsymbol{p})|0\rangle \\
&= \int \tilde{d\boldsymbol{k}} \tilde{d\boldsymbol{p}} \Psi_1(\boldsymbol{p}) a^\dagger(\boldsymbol{k}) \underbrace{a(\boldsymbol{k}) a^\dagger(\boldsymbol{p})}_{\xi a^\dagger(\boldsymbol{p}) a(\boldsymbol{k}) + (2\pi)^3 2\omega_{\boldsymbol{p}} \delta(\boldsymbol{k}-\boldsymbol{p})} |0\rangle \\
&= \int \tilde{d\boldsymbol{k}} \tilde{d\boldsymbol{p}} \Psi_1(\boldsymbol{p}) a^\dagger(\boldsymbol{k}) (2\pi)^3 2\omega_{\boldsymbol{p}} \delta(\boldsymbol{k}-\boldsymbol{p})|0\rangle \\
&= \int \tilde{d\boldsymbol{p}} \Psi_1(\boldsymbol{p}) a^\dagger(\boldsymbol{p})|0\rangle = |\Psi_1\rangle
\end{aligned}$$

B. Quantum field

1. **Linear superposition** parameterized by a formal $\boldsymbol{x}$ variable

$$\phi(\boldsymbol{x}) = \int \frac{d^3p}{(2\pi)^3 2\omega(\boldsymbol{p})} a(\boldsymbol{p}) e^{i\boldsymbol{x}\boldsymbol{p}} = \int \tilde{d\boldsymbol{p}} a(\boldsymbol{p}) e^{i\boldsymbol{x}\boldsymbol{p}}$$

2. **Dynamics:** Intuitively

$$\phi(t, \boldsymbol{x}) = \int \tilde{d\boldsymbol{p}} a(\boldsymbol{p}) e^{-iE(\boldsymbol{p})t + i\boldsymbol{x}\boldsymbol{p}}.$$

3. **Anti-particles:** two sets of boxes

$$\begin{aligned}
[a(\boldsymbol{p}), a^\dagger(\boldsymbol{q})]_\xi &= [b(\boldsymbol{p}), b^\dagger(\boldsymbol{q})]_\xi = (2\pi)^3 2\omega(\boldsymbol{p}) \delta(\boldsymbol{p}-\boldsymbol{q}) \\
[a(\boldsymbol{p}), a(\boldsymbol{q})]_\xi &= [a^\dagger(\boldsymbol{p}), a^\dagger(\boldsymbol{q})]_\xi = [b(\boldsymbol{p}), b(\boldsymbol{q})]_\xi = [b^\dagger(\boldsymbol{p}), b^\dagger(\boldsymbol{q})]_\xi = [a(\boldsymbol{p}), b(\boldsymbol{q})]_\xi = [a^\dagger(\boldsymbol{p}), b^\dagger(\boldsymbol{q})]_\xi = 0
\end{aligned}$$

$$\boxed{\phi(t, \boldsymbol{x}) = \int \frac{d^3p}{(2\pi)^3 2\omega(\boldsymbol{p})} [a(\boldsymbol{p}) e^{-iE(\boldsymbol{p})t + i\boldsymbol{x}\boldsymbol{p}} + b^\dagger(\boldsymbol{p}) e^{iE(\boldsymbol{p})t - i\boldsymbol{x}\boldsymbol{p}}] = \int \tilde{d^3p} [a(\boldsymbol{p}) e^{-ixp} + b^\dagger(\boldsymbol{p}) e^{ixp}]_{|p^0 = \omega_{\boldsymbol{p}}}}$$

4. **General solution of Klein-Gordon equation:**

- E.O.M.: dispersion relation

(a) Schrödinger equation: translation in time $H(p) = \frac{\boldsymbol{p}^2}{2m}$, $\boldsymbol{p} = -i\boldsymbol{\partial}$

$$i\partial_t \psi = -\frac{\Delta}{2m} \psi$$

(b) Klein-Gordon equation: $p^2 = m^2$, $p^\mu = i\partial^\mu$

$$(\Box + m^2)\phi(x) = 0$$

- General solution:

$$\begin{aligned}
\phi(x) &= \int \frac{d^4p}{(2\pi)^4} f(p) e^{-ipx} \\
(\Box + m^2)\phi(x) &= \int \frac{d^4p}{(2\pi)^4} f(p)(m^2 - p^2) e^{ipx} \\
f(p)(m^2 - p^2) &= 0 \quad \Longrightarrow \quad f(p) = g(p) 2\pi\delta(p^2 - m^2)
\end{aligned}$$

- Separation of particles and anti-particles: $\omega_{\boldsymbol{p}} = \sqrt{m^2 + \boldsymbol{p}^2}$, $\Theta(x) = \begin{cases} 1 & x > 0 \\ \text{ill defined} & x = 0 \\ 0 & x < 0 \end{cases}$

$$\begin{aligned}
a(\boldsymbol{p}) &= g(\omega_{\boldsymbol{p}}, \boldsymbol{p}), \quad b^\dagger(-\boldsymbol{p}) = g(-\omega_{\boldsymbol{p}}, \boldsymbol{p}) \\
\phi(t, \boldsymbol{x}) &= \int \frac{d^4p}{(2\pi)^4} 2\pi\delta(p^2 - m^2)\Theta(p^0)[a(\boldsymbol{p})e^{-ip^0t + i\boldsymbol{x}\boldsymbol{p}} + b^\dagger(-\boldsymbol{p})e^{ip^0t + i\boldsymbol{x}\boldsymbol{p}}] \\
&= \int \frac{d^4p}{(2\pi)^4} 2\pi\delta(p^2 - m^2)\Theta(p^0)[a(\boldsymbol{p})e^{-ip^0t + i\boldsymbol{x}\boldsymbol{p}} + b^\dagger(\boldsymbol{p})e^{ip^0t - i\boldsymbol{x}\boldsymbol{p}}]
\end{aligned}$$

- Dirac-delta:

$$\begin{aligned}
\int dx \delta(x - x_0) f(x) &= f(x_0) \\
\int dx \delta(h(x)) f(x) &= \int dh \left| \frac{dx}{dh} \right| \delta(h) f(x(h)) = \sum_{x'} \frac{f(x')}{|h'(x')|}, \quad h(x') = 0
\end{aligned}$$

- Final form:

$$\begin{aligned}
\phi(t, \boldsymbol{x}) &= \int \frac{d^3p}{(2\pi)^3 2\omega_{\boldsymbol{p}}} [a(\boldsymbol{p})e^{-i\omega_{\boldsymbol{p}}t + i\boldsymbol{x}\boldsymbol{p}} + b^\dagger(\boldsymbol{p})e^{i\omega_{\boldsymbol{p}}t - i\boldsymbol{x}\boldsymbol{p}}] \\
&= \int \tilde{d\boldsymbol{p}} [a(\boldsymbol{p})e^{-i\omega_{\boldsymbol{p}}t + i\boldsymbol{x}\boldsymbol{p}} + b^\dagger(\boldsymbol{p})e^{i\omega_{\boldsymbol{p}}t - i\boldsymbol{x}\boldsymbol{p}}] \\
&= \int \tilde{d\boldsymbol{p}} [a(\boldsymbol{p})e^{-ipx} + b^\dagger(\boldsymbol{p})e^{ipx}]_{|p^0 = \omega_{\boldsymbol{p}}}
\end{aligned}$$

- The integration measure

$$\tilde{d\boldsymbol{k}} = \frac{d^4k}{(2\pi)^4} 2\pi\delta(k^2 - m^2)\Theta(k^0)_{|k^0 = \omega_{\boldsymbol{k}}} = \frac{d^3k}{(2\pi)^3 2\omega_{\boldsymbol{k}}}$$

is invariant under the proper Lorentz group $\Longrightarrow$ $a(\boldsymbol{k})$, $b(\boldsymbol{k})$ are scalars

5. **A little history:**

(a) $\phi(t, \boldsymbol{x})$ is like the wave function of a free particle

(b) $\phi(t, \boldsymbol{x})$ is an operator

(c) c-number is replaced by operator $\Longrightarrow$ (second) quantization

(d) Today: Wrong interpretation, there is no single particle wave function for interacting many-body systems.

C. Representations of time dependence

1. **Schrödinger representation:** State: time-dependent, observables: time-independent

   - E.O.M.:

$$i\partial_t|\psi(t)\rangle_S = H|\psi(t)\rangle_S, \quad i\partial_t A_S = 0$$

   - Solution:

$$|\psi(t)\rangle_S = e^{-i(t-t_i)H}|\psi(t_i)\rangle_S.$$

2. **Heisenberg representation:** State: time-independent, observables: time-dependent

   - States:

$$|\Psi(t)\rangle_H = e^{i(t-t_i)H}|\Psi(t)\rangle_S$$

   - Operators: Conservation of the matrix elements:

$$\begin{aligned}
\langle A(t)\rangle &= \underbrace{\langle\Psi(t_i)|e^{iH(t-t_i)}}_{\langle\Psi(t)|_S} A_S \underbrace{e^{-iH(t-t_i)}|\Psi(t_i)\rangle}_{|\Psi(t)\rangle_S} = \langle\Psi(t_i)|\underbrace{e^{iH(t-t_i)}A_S e^{-iH(t-t_i)}}_{A_H(t)}|\Psi(t_i)\rangle \\
A_H(t) &= e^{i(t-t_i)H}A_S e^{-i(t-t_i)H}
\end{aligned}$$

   - E.O.M.:

$$\begin{aligned}
i\partial_t|\Psi\rangle_H &= 0, \quad |\psi_H\rangle = |\psi(t_i)\rangle_S \\
i\partial_t A_H(t) &= [A_H(t), H], \quad A_H(t_i) = A_S
\end{aligned}$$

3. **Interaction representation:** $H = H_0 + H_1$,

   - observables: time-dependent by H_0 (perturbation expansion would be complicated)

   - state: time-independent by H_1

- Complication $e^{-it(H_0+H_1)} \neq e^{-itH_0}e^{-itH_1}$

$$\begin{aligned} |\Psi(t)\rangle_i &= e^{i(t-t_i)H_0}|\Psi(t)\rangle_S \\ A_i(t) &= e^{i(t-t_i)H_0}A_S e^{-i(t-t_i)H_0} \end{aligned}$$

E.O.M.:

$$\begin{aligned} i\partial_t|\Psi(t)\rangle_i &= -H_0|\Psi(t)\rangle_i + e^{i(t-t_i)H_0}(H_0+H_1)|\Psi(t)\rangle_S \\ &= -H_0|\Psi(t)\rangle_i + e^{i(t-t_i)H_0}(H_0+H_1)e^{-i(t-t_i)H_0}e^{i(t-t_i)H_0}|\Psi(t)\rangle_S \\ &= H_{1i}(t)|\Psi(t)\rangle_i \\ i\partial_t A_i(t) &= [A_i, H_0] \end{aligned}$$

4. **Schrödinger's equation with time dependent Hamiltonian:**

$$i\partial_t|\Psi(t)\rangle = H(t)|\Psi(t)\rangle.$$

- *For C-numbers:*

$$\begin{aligned} i\partial_t\Psi(t) &= H(t)\Psi(t) \\ \Psi(t) &= e^{-i\int_{t_i}^t dt' H(t')}\Psi(t_i) \end{aligned}$$

  Proof:

$$\begin{aligned} \Psi(t+\Delta t) &= e^{-i\Delta t\sum_{j=1}^{N+1}H(t_j)}\Psi(t_i) \\ &= e^{-i\Delta t H(t_{N+1})}\underbrace{e^{-i\Delta t\sum_{j=1}^{N}H(t_j)}\Psi(t_i)}_{\Psi(t)} \\ &\approx [1-i\Delta t H(t_N)]\Psi(t) \\ i\frac{\Psi(t+\Delta t)-\Psi(t)}{\Delta t} &= H(t)\Psi(t) \end{aligned}$$

- *Operators:*

$$e^{-i\Delta t\sum_{j=1}^{N+1}H(t_j)} \neq e^{-i\Delta t H(t_{N+1})}e^{-i\Delta t\sum_{j=1}^{N}H(t_j)}$$

- *Time ordered (chronological) product:* $\xi = \begin{cases} 1 & \text{bosonic operators} \\ -1 & \text{bosonic operators} \end{cases}$

$$T[A(t_A)B(t_B)] = \Theta(t_A-t_B)A(t_A)B(t_B) + \xi\Theta(t_B-t_A)B(t_B)A(t_A)$$

- *Solution:*

$$|\Psi(t)\rangle = U(t,t_i)|\Psi(t_i)\rangle, \quad U(t,t_i) = T[e^{-i\int_{t_i}^t dt' H(t')}]$$

Proof:

$$\begin{aligned}
i\partial_t U(t,t_i)|\Psi(t_i)\rangle &= H(t)U(t,t_i)|\Psi(t_i)\rangle \\
i\partial_t U(t,t_i) &= H(t)U(t,t_i) \\
&= i\partial_t T[e^{-i\int_{t_i}^t dt' H(t')}] \\
&= i\partial_t \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_i}^t dt_1 \cdots \int_{t_i}^t dt_n T[H(t_1)\cdots H(t_n)] \\
&= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} n \int_{t_i}^t dt_2 \cdots \int_{t_i}^t dt_n T[H(t)H(t_1)\cdots H(t_{n-1})] \\
&= H(t) \sum_{n=0}^{\infty} \frac{(-i)^{n-1}}{(n-1)!} \int_{t_i}^t dt_1 \cdots \int_{t_i}^t dt_{n-1} T[H(t_1)\cdots H(t_{n-1})] \\
&= H(t)U(t,t_i).
\end{aligned}$$

5. **Fields:**

- Undetermined time dependence:

$$\phi(t,\boldsymbol{x}) = \int \tilde{d\boldsymbol{p}}[a(\boldsymbol{p})e^{-ipx} + b^\dagger(\boldsymbol{p})e^{ipx}] \rightarrow \int \tilde{dp}[a(t,\boldsymbol{q})e^{\boldsymbol{i x p}} + b^\dagger(t,\boldsymbol{p})e^{-\boldsymbol{i x p}}]$$

- *Hamiltonian:*

$$H = \int \tilde{dp}\omega_{\boldsymbol{p}} \left[a^\dagger(t,\boldsymbol{p})a(t,\boldsymbol{p}) + b^\dagger(t,\boldsymbol{p})b(t,\boldsymbol{p}) + 1\right]$$

(H is time-independent hence we may take the creation and destruction operators at arbitrary time)

- *Heisenberg equation:*

$$i\partial_0 a(t,\boldsymbol{q}) = [a(t,\boldsymbol{q}),H] = \int \frac{d^3p}{(2\pi)^3 2\omega_{\boldsymbol{p}}} \omega(\boldsymbol{p})[a(t,\boldsymbol{q}), a^\dagger(t,\boldsymbol{p})a(t,\boldsymbol{p})]_+$$

Useful identity: $[A,BC]_+ = ABC - BCA = ABC - BCA + BAC - BAC = B[A,C]_+ + [A,B]_+ C$

$$\begin{aligned}
i\partial_0 a(t,\boldsymbol{q}) &= \int \frac{d^3p}{(2\pi)^3 2\omega_{\boldsymbol{p}}} \omega(\boldsymbol{p}) \{a^\dagger(t,\boldsymbol{p}) \underbrace{[a(t,\boldsymbol{q}), a(t,\boldsymbol{p})]}_{0} + \underbrace{[a(t,\boldsymbol{q}), a^\dagger(t,\boldsymbol{p})]}_{(2\pi)^3 2\omega_{\boldsymbol{p}}\delta(\boldsymbol{p}-\boldsymbol{q})} a(t,\boldsymbol{p})\} \\
&= \omega_{\boldsymbol{q}} a(t,\boldsymbol{q}) \\
a(t,\boldsymbol{q}) &= e^{-i(t-t_i)\omega_{\boldsymbol{q}}} a(t_i,\boldsymbol{q})
\end{aligned}$$

II. CLASSICAL FIELD THEORY

A. Variational principle

Invariance of the equation of motion under non-linear, time-dependent change of coordinates

1. **Single point on the real axis:**

   - *Problem:* identification of $x_{cl} \in \mathbb{R}$ in a reparametrization independent manner

     Solution: Find a function with vanishing derivative only at x_{cl}, $\frac{df(x)}{dx}_{|x=x_{cl}} = 0$

     Proof: new coordinate y,

$$\frac{df(x(y))}{dy}_{|y=y_{cl}} = \underbrace{\frac{df(x)}{dx}_{|x=x_{cl}}}_{0} \frac{dx(y)}{dy}_{|y=y_{cl}} = 0$$

   - *Variational principle:* infinitesimal variation $x \to x + \delta x$,

$$\begin{aligned} f(x_{cl} + \delta x) &= f(x_{cl}) + \delta f(x_{cl}) \\ &= f(x_{cl}) + \delta x \underbrace{f'(x_{cl})}_{0} + \frac{\delta x^2}{2} f''(x_{cl}) + \mathcal{O}\left(\delta x^3\right) \\ \delta f(x_{cl}) &= \mathcal{O}\left(\delta x^2\right) \end{aligned}$$

2. **Non-relativistic point particle:**

   - *Problem:* identification of $x(t)$ in a coordinate choice independent manner.

   - *Variational principle:*

     (a) identify a trajectory $x_{cl}(t)$ by $x_{cl}(t_i) = x_i$, $x_{cl}(t_f) = x_f$

     (b) variation: $x(t) \to x(t) + \delta x(t)$, $\delta x(t_i) = \delta x(t_f) = 0$

     (c) action:

$$f(x) \to S[x] = \int_{t_i}^{t_f} dt \underbrace{L(x(t), \dot{x}(t))}_{\text{Lagrangian}}$$

   - *E.O.M.:*

     (a) variation:

$$\begin{aligned} \delta S[x] &= \int_{t_i}^{t_f} dt L\left(x(t) + \delta x(t), \dot{x}(t) + \delta \frac{d}{dt} x(t)\right) - \int_{t_i}^{t_f} dt L(x(t), \dot{x}(t)) \\ &= \int_{t_i}^{t_f} dt \Bigg[L(x(t), \dot{x}(t)) + \delta x(t) \frac{\partial L(x(t), \dot{x}(t))}{\partial x} + \underbrace{\delta \frac{d}{dt} x(t)}_{\frac{d}{dt}\delta x(t)} \frac{\partial L(x(t), \dot{x}(t))}{\partial \dot{x}} + \mathcal{O}\left(\delta x(t)^2\right) \\ &\quad - \int_{t_i}^{t_f} dt L(x(t), \dot{x}(t)) \Bigg] \\ &= \int_{t_i}^{t_f} dt \delta x(t) \left[\frac{\partial L(x(t), \dot{x}(t))}{\partial x} - \frac{d}{dt} \frac{\partial L(x(t), \dot{x}(t))}{\partial \dot{x}} \right] + \underbrace{\delta x(t)}_{0} \frac{\partial L(x(t), \dot{x}(t))}{\partial \dot{x}} \Bigg|_{t_f}^{t_i} + \mathcal{O}\left(\delta x(t)^2\right) \end{aligned}$$

     (b) Euler-Lagrange equation:

$$0 = \frac{\partial L(x, \dot{x})}{\partial x} - \frac{d}{dt} \frac{\partial L(x, \dot{x})}{\partial \dot{x}}$$

(c) n-dimensional particle:

$$0 = \frac{\partial L(\boldsymbol{x}, \dot{\boldsymbol{x}})}{\partial \boldsymbol{x}} - \frac{d}{dt}\frac{\partial L(\boldsymbol{x}, \dot{\boldsymbol{x}})}{\partial \dot{\boldsymbol{x}}}$$

- Lagrangian:

$$L = T - U = \frac{m}{2}\dot{\boldsymbol{x}}^2 - U(\boldsymbol{x}) \quad \rightarrow \quad m\ddot{\boldsymbol{x}} = -\boldsymbol{\nabla} U(\boldsymbol{x})$$

- generalized momentum: $p = \frac{\partial L(x,\dot{x})}{\partial \dot{x}} \quad \Longrightarrow \quad \dot{p} = \frac{\partial L(x,\dot{x})}{\partial x}$
- cyclic coordinate: $\frac{\partial L(x,\dot{x})}{\partial x_{cycl}} = 0$
- There is a conserved quantity, p_{cycl}, for each cyclic coordinate

3. **Scalar field:**

- *Problem:* identification of $\phi_a(x)$, $a = 1, \ldots, n$
- *Variational principle:* $\phi(x) \rightarrow \phi(x) + \delta\phi(x)$, $\delta\phi(t_i, \boldsymbol{x}) = \delta\phi(t_f, \boldsymbol{x}) = 0$
- *Variation:*

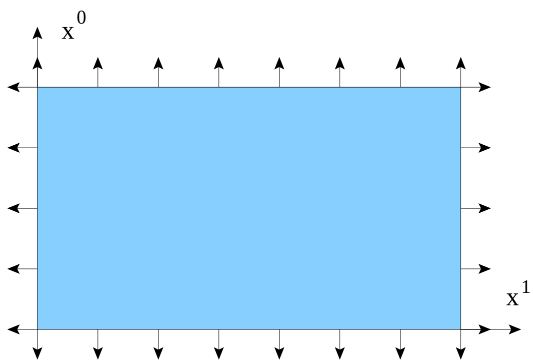
$$S[\phi] = \int_V \underbrace{dt d^3x}_{dx} L(\phi, \partial\phi)$$

- *E.O.M.:*

$$\begin{aligned}
\delta S &= \int_V dx \left(\frac{\partial L(\phi, \partial\phi)}{\partial \phi_a}\delta\phi_a + \frac{\partial L(\phi, \partial\phi)}{\partial \partial_\mu \phi_a}\delta\partial_\mu\phi_a \right) + \mathcal{O}\left(\delta^2\phi\right) \\
&= \int_V dx \left(\frac{\partial L(\phi, \partial\phi)}{\partial \phi_a}\delta\phi_a + \frac{\partial L(\phi, \partial\phi)}{\partial \partial_\mu \phi_a}\partial_\mu\delta\phi_a \right) + \mathcal{O}\left(\delta^2\phi\right) \\
&= \int_{\partial V} ds^\mu \delta\phi_a \frac{\partial L(\phi, \partial\phi)}{\partial \partial_\mu \phi_a} + \int_V dx \delta\phi_a \left(\frac{\partial L(\phi, \partial\phi)}{\partial \phi_a} - \partial_\mu \frac{\partial L(\phi, \partial\phi)}{\partial \partial_\mu \phi_a} \right) + \mathcal{O}\left(\delta^2\phi\right)
\end{aligned}$$

$\nwarrow$

0 (local Lagrangian) Euler-Lagrange equation:



$$\frac{\partial L(\phi, \partial\phi)}{\partial \phi_a} - \partial_\mu \frac{\partial L(\phi, \partial\phi)}{\partial \partial_\mu \phi_a} = 0.$$

- *Current* associated to the field ϕ:

$$j^\mu_\phi = \frac{\partial L}{\partial \partial \phi_\mu}$$

- *Conservation law:* Current of a cyclic field variable is conserved

$$\boxed{\partial_\mu j^\mu_{\phi_{cycl}} = 0}$$

- *Examples:*

(a) *Scalar particle:*

$$L = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - U(\phi) \quad \to \quad (\partial_\mu \partial^\mu + m^2) = -U'(\phi)$$

Normal modes:

i. Fourier transformation in space

$$\begin{aligned}
\phi(x) &= \int \frac{d^3k}{(2\pi)^3}\phi_{\boldsymbol{k}}(x^0)e^{i\boldsymbol{k}\boldsymbol{x}} \\
S &= \int dx \frac{d^3k d^3q}{(2\pi)^6}\left[\frac{1}{2}\partial_0\phi_{\boldsymbol{k}}\partial_0\phi_{\boldsymbol{q}} + \frac{\boldsymbol{k}\boldsymbol{q}}{2}\phi_{\boldsymbol{k}}\phi_{\boldsymbol{q}} - \frac{m^2c^2}{2\hbar^2}\phi_{\boldsymbol{k}}\phi_{\boldsymbol{q}}\right]e^{i(\boldsymbol{k}+\boldsymbol{q})\boldsymbol{x}} \\
&= \int dx^0 \frac{d^3k}{(2\pi)^3}\left[\frac{1}{2}\partial_0\phi_{-\boldsymbol{k}}\partial_0\phi_{\boldsymbol{k}} - \frac{1}{2}\left(\boldsymbol{k}^2 + \frac{m^2c^2}{\hbar^2}\right)\phi_{-\boldsymbol{k}}\phi_{\boldsymbol{k}}\right] \quad \leftarrow \quad \delta(\boldsymbol{x}) = \int \frac{d^3k}{(2\pi)^3}e^{i\boldsymbol{k}\boldsymbol{x}}
\end{aligned}$$

ii. Real field:

$$\begin{aligned}
\phi(x) &= \int \frac{d^3k}{(2\pi)^3}\phi_{\boldsymbol{k}}(x^0)e^{i\boldsymbol{k}\boldsymbol{x}} = \phi^*(x) = \int \frac{d^3k}{(2\pi)^3}\phi^*_{\boldsymbol{k}}(x^0)e^{-i\boldsymbol{k}\boldsymbol{x}} \\
\Longrightarrow \quad \phi_{-\boldsymbol{k}}(x^0) &= \phi^*_{\boldsymbol{k}}(x^0) \\
S &= \int dx^0 \frac{d^3k}{(2\pi)^3}\left[\frac{1}{2}\partial_0\phi^*_{\boldsymbol{k}}\partial_0\phi_{\boldsymbol{k}} - \frac{1}{2}\left(\boldsymbol{k}^2 + \frac{m^2c^2}{\hbar^2}\right)\phi^*_{\boldsymbol{k}}\phi_{\boldsymbol{k}}\right]
\end{aligned}$$

iii. Relativistic normal modes: plane waves, $\phi_{\boldsymbol{k}}(x^0) = \phi_{1,\boldsymbol{k}}(x^0) + i\phi_{2,\boldsymbol{k}}(x^0)$

$$\begin{aligned}
S[\phi_{\boldsymbol{k}}] &= \sum_{j=1,2}\int dx^0 \left[\frac{1}{2}[\partial_0\phi_{j,\boldsymbol{k}}(x^0)]^2 - \frac{1}{2}\left(\boldsymbol{k}^2 + \frac{m^2c^2}{\hbar^2}\right)[\phi_{j,\boldsymbol{k}}(x^0)]^2\right] \\
&= \sum_{j=1,2}\int dx^0 \left[\frac{M_{\boldsymbol{k}}}{2}[\partial_t\phi_{j,\boldsymbol{k}}(x^0)]^2 - \frac{M_{\boldsymbol{k}}\Omega^2_{\boldsymbol{k}}}{2}\phi_{j,\boldsymbol{k}}(x^0)\phi_{j,\boldsymbol{k}}(x^0)\right] \\
M_{\boldsymbol{k}} &= \frac{1}{c^2}, \quad \Omega^2_{\boldsymbol{k}} = c^2\left(\boldsymbol{k}^2 + \frac{m^2c^2}{\hbar^2}\right), \quad E_{\boldsymbol{k}} = \hbar\Omega_{\boldsymbol{k}} \\
p &= \left(\frac{E}{c}, \boldsymbol{p}\right) = \left(\frac{\hbar c\sqrt{\boldsymbol{k}^2 + \frac{m^2c^2}{\hbar^2}}}{c}, \hbar\boldsymbol{k}\right) = (\sqrt{m^2c^2 + \boldsymbol{p}^2}, \boldsymbol{p})
\end{aligned}$$

(b) *Free fermions:*

$$\begin{aligned}
L &= \bar\psi[i\partial_\mu\gamma^\mu - m]\psi \to \frac{i}{2}[\bar\psi\gamma^\mu(\partial_\mu\psi) - (\partial_\mu\bar\psi)\gamma^\mu\psi] - m\bar\psi\psi \quad \to \quad (i\partial_\mu\gamma^\mu - m)\psi(x) = 0 \\
\gamma^0 &= \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \gamma^\ell = \begin{pmatrix} 0 & \sigma^\ell \\ -\sigma^\ell & 0 \end{pmatrix}
\end{aligned}$$

(c) *Yukawa model (proton + σ meson):*

$$L = \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \frac{M^2}{2}\sigma^2 + \bar{\psi}[i\partial_\mu\gamma^\mu - m - g\sigma]\psi$$

B. Noether theorem

1. **There is a conserved current for each continuous symmetry.**

- *Symmetry:* $x^\mu \to x'^\mu$, $\phi_a(x) \to \phi'_a(x)$ and preserves the E.O.M.,

$$L(\phi, \partial\phi) \to L(\phi', \partial'\phi') + \partial'_\mu\Lambda^\mu$$

- *Continuous symmetry:*

  infinitesimal symmetry transformations, $x^\mu \to x^\mu + \delta x^\mu$, $\phi_a(x) \to \phi_a(x) + \delta\phi_a(x)$

  Examples: rotations, translations, $\phi(x) \to e^{i\alpha}\phi(x)$

- *Conserved current:* $\partial_\mu j^\mu = 0$, conserved charge: $Q(t)$:

$$\partial_0 Q(t) = \partial_0 \int_V d^3x j^0 = -\int_V d^3x \boldsymbol{\partial} v j = -\int_{\partial V} d\boldsymbol{s}\cdot\boldsymbol{j}$$

- *External and internal spaces:*

$$\phi_a(x): \underbrace{\mathbb{R}^4}_{\text{external space}} \to \underbrace{\mathbb{R}^m}_{\text{internal space}}.$$

2. **Continuous groups:**

- $\{\omega(\alpha)\}$:

  (a) continuous topology (infinitesimal neighborhoods)

  (b) multiplication law:

$$\omega(\alpha)\omega(\beta) = \omega(F(\alpha, \beta))$$

  Convention: $\omega(0) = \mathbb{1}$

  (c) Examples: translations, $\alpha = x^\mu$, $F(\alpha, \beta) = \alpha + \beta$

  (d) Ado's theorem: any finite dimensional Lie-algebra is identical with a subspace of the generators of the matrix group $GL(N)$, with sufficiently large N

- *Infinitesimal group elements:*

$$\omega = \mathbb{1} + \sum_{n=1}^{n} \epsilon^a\tau^a + \mathcal{O}\left(\epsilon^2\right),$$

- *Generators:* $\tau^a = \frac{\partial\omega(0)}{\partial\alpha^a}$
- *Exponential map:*

$$\begin{aligned}\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n &= \lim_{n\to\infty} e^{n\ln(1+\frac{a}{n})} = \lim_{n\to\infty} e^{n\left(\frac{a}{n}+\mathcal{O}\left(n^{-2}\right)\right)} = e^a \\ e^{\sum_a \alpha^a\tau^a} &= \lim_{n\to\infty}\left(1+\sum_a \frac{\alpha^a}{n}\tau^a\right)^n\end{aligned}$$

  Any group element can be obtained in such a form in a connected group

- *Lie-algebra:*

$$[\tau^a, \tau^b] = \sum_c f^{a,b,c}\tau^c.$$

  Structure constants: $f^{a,b,c}$, uniquely determine the multiplication of infinitesimal group elements

TABLE I: Real classical matrix groups.

Symbol	Name	Definition	Dimension	Generators
$GL(N)$	general linear group	$\det A \neq 0^a$	N^2	$\{\tau$: real $N\times N$ matrices$\}$
$SL(N)$	special linear group	$\det A = 1$	N^2-1	$\mathrm{tr}\tau = 0^b$
$O(N)$	orthogonal group	$A^{tr}A = \mathbb{1}^c$	$\frac{1}{2}N(N-1)$	$\tau^{tr} = -\tau$
$SO(N)$	special orthogonal group	$A^{tr}A = \mathbb{1}$, $\det A = 1$	$\frac{1}{2}N(N-1)$	$\tau^{tr} = -\tau$, $\mathrm{tr}\tau = 0$

[a]The matrix A is supposed to be an element of the group in question.
[b]$\det(\mathbb{1}+\epsilon\tau) = 1+\epsilon\mathrm{tr}\tau+\mathcal{O}\left(\epsilon^2\right)$
[c]$\det A^{tr}A = (\det A)^2 = 1$ and $\det A = \pm 1$.

TABLE II: Complex classical matrix groups.

Symbol	Name	Definition	Dimension	Generators
$GL(N,C)$	complex general linear group	$\det A \neq 0$	$2N^2$	$\{\tau$: complex $N\times N$ matrices$\}$
$SL(N,C)$	complex special linear group	$\det A = 1$	$2N^2-2$	$\mathrm{tr}\tau = 0$
$U(N)$	unitary group	$A^\dagger A = \mathbb{1}^a$	N^2	$\tau^\dagger = -\tau$
$SU(N)$	special unitary group	$A^\dagger A = \mathbb{1}$, $\det A = 1$	N^2-1	$\tau^\dagger = -\tau$, $\mathrm{tr}\tau = 0$

[a]$\det A^\dagger A = (\det A)^* \det A = |\det A|^2 = 1$

3. **Internal symmetries:** $\delta x^\mu = 0$

- *Linear symmetry:*

$$\begin{aligned}\delta\phi_a(x) &= \epsilon \underbrace{\tau_{ab}}_{generator} \phi_b(x) \\ L(\phi,\partial\phi) &= L(\phi+\epsilon\tau\phi, \partial\phi+\epsilon\tau\partial\phi) + \mathcal{O}\left(\epsilon^2\right) \\ 0 &= \epsilon\tau\frac{\partial L(\phi,\partial\phi)}{\partial\phi} + \epsilon\tau\partial\phi\frac{\partial L(\phi,\partial\phi)}{\partial\partial\phi}\end{aligned}$$

- *$\phi(x)$ satisfies the E.O.M.:*

$$0 = \frac{\partial L(\phi, \partial\phi)}{\partial\phi} - \partial_\mu \frac{\partial \tilde{L}(\phi, \partial\phi)}{\partial\partial_\mu\phi}$$

- *Arbitrary variation:*

$$\delta S = \mathcal{O}\left(\delta^2\phi\right)$$

- *Special variation:* $\phi(x) \to \phi(x) + \epsilon(x)\tau\phi(x)$,
- *Linearized Lagrangian:*

$$\begin{aligned}\tilde{L}(\epsilon, \partial\epsilon) &= L(\phi + \epsilon\tau\phi(x), \partial\phi + \partial\epsilon\tau\phi + \epsilon\tau\partial\phi) \\ &\to \epsilon\tau\frac{\partial L(\phi, \partial\phi)}{\partial\phi} + (\partial\epsilon\tau\phi + \epsilon\tau\partial\phi)\frac{\partial L(\phi, \partial\phi)}{\partial\partial\phi}\end{aligned}$$

- *Cyclic coordinate:*

$$\delta S = \mathcal{O}\left(\epsilon^2\right) \quad \to \quad \frac{\partial \tilde{L}(\epsilon, \partial\epsilon)}{\partial\epsilon} - \partial_\mu \frac{\partial \tilde{L}(\epsilon, \partial\epsilon)}{\partial\partial_\mu\epsilon} = 0$$

- *Noether current:*

$$J^\mu = -\frac{\partial \tilde{L}(\epsilon, \partial\epsilon)}{\partial\partial_\mu\epsilon} = -\frac{\partial L(\phi, \partial\phi)}{\partial\partial_\mu\phi}\tau\phi$$

  is conserved N.B.:

  (a) there is an independent conserved current corresponding to each independent direction in the internal symmetry group

  (b) the conserved current is well defined up to a multiplicative constant

- *Examples:*

  (a) n-component real scalar field: ϕ_a, $a = 1, \cdots, n$, $G = O(n)$,

$$\begin{aligned}L &= \frac{1}{2}(\partial\phi)^2 - V(\phi^2) \\ \delta\phi &= \epsilon^a\tau^a\phi \\ J^a_\mu &= -\partial_\mu\phi\tau^a\phi\end{aligned}$$

  (b) Single complex scalar field: $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$, $G = U(1)$, $\phi(x) \to e^{i\alpha}\phi(x)$

$$\begin{aligned}L &= \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 - \frac{m^2}{2}(\phi_1^2 + \phi_2^2) - V\left(\frac{1}{2}(\phi_1^2 + \phi_2^2)\right) \\ &= \partial_\mu\phi^*\partial^\mu\phi + \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^\dagger\phi - V(\phi^\dagger\phi)\end{aligned}$$

  Field variable:

i. $\binom{\phi}{\phi^*}$:

$$\begin{pmatrix}\phi\\ \phi^*\end{pmatrix} : \begin{pmatrix}\phi\\ \phi^*\end{pmatrix} \to \begin{pmatrix}e^{i\alpha}\phi\\ e^{-i\alpha}\phi^*\end{pmatrix}, \quad \delta\begin{pmatrix}\phi\\ \phi^*\end{pmatrix} = i\alpha\begin{pmatrix}\phi\\ -\phi^*\end{pmatrix} = \alpha\tau\begin{pmatrix}\phi\\ \phi^*\end{pmatrix}, \quad \tau = i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$J = -\frac{\partial L}{\partial\partial_\mu\phi}\tau\phi = -i\left(\frac{\partial L}{\partial\partial_\mu\phi}\phi - \frac{\partial L}{\partial\partial_\mu\phi^*}\phi^*\right) = -i(\partial_\mu\phi^*\phi - \phi^*\partial_\mu\phi) = i\phi^*\overleftrightarrow{\partial}_\mu\phi$$

ii. $\binom{\phi_1}{\phi_2}$:

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} : \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \to e^{i\alpha}\phi = \frac{1}{\sqrt{2}}[\cos\alpha\phi_1 - \sin\alpha\phi_2 + i(\cos\alpha\phi_2 + \sin\alpha\phi_1)]$$

$$\delta\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \alpha\begin{pmatrix}-\phi_2\\ \phi_1\end{pmatrix} = \alpha\tau\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \quad \tau = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

$$\begin{aligned} J &= -\frac{\partial L(\phi, \partial\phi_1)}{\partial\partial_\mu\phi}\tau\phi = -\left(-\frac{\partial L}{\partial\partial_\mu\phi_1}\phi_2 + \frac{\partial L}{\partial\partial_\mu\phi_2}\phi_1\right) = \partial_\mu\phi_1\phi_2 - \partial_\mu\phi_2\phi_1 \\ &= \frac{i}{2}[\partial_\mu(\phi_1+i\phi_2)^*(\phi_1+i\phi_2) - (\phi_1+i\phi_2)^*\partial_\mu(\phi_1+i\phi_2)] = -i(\partial_\mu\phi^*\phi - \phi^*\partial_\mu\phi) \end{aligned}$$

(c) n-component complex scalar field: ϕ_a, $a = 1, \cdots, n$, $G = U(n)$

$$\begin{aligned} L &= \partial\phi^\dagger\partial\phi - V(\phi^\dagger\phi) \\ \delta\phi &= \epsilon^a\tau^a\phi, \quad \delta\phi^\dagger = \epsilon^a(\phi\tau^a)^\dagger = -\epsilon^a\phi^\dagger\tau^a \\ J^a_\mu &= -\partial_\mu\phi^\dagger\tau^a\phi + \partial_\mu\phi(\tau^a)^{\mathrm{tr}}\phi^\dagger = -\partial_\mu\phi^\dagger\tau^a\phi + \phi^\dagger\tau^a\partial_\mu\phi = \phi^\dagger\tau^a\overleftrightarrow{\partial}_\mu\phi \end{aligned}$$

(d) Electron: ψ, $G = U(1)$, $\psi \to e^{i\alpha}\psi$, $\bar\psi \to e^{-i\alpha}\bar\psi$, $\tau = i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

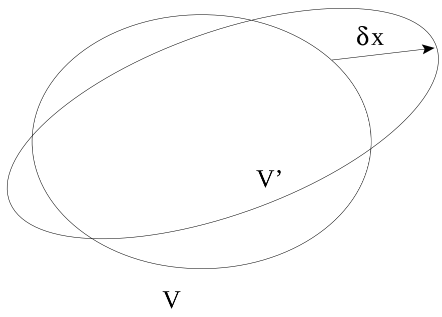
$$\begin{aligned} L &= \frac{i}{2}[\bar\psi\gamma^\mu(\partial_\mu\psi) - (\partial_\mu\bar\psi)\gamma^\mu\psi] - m\bar\psi\psi \\ J_\mu &= \frac{1}{2}\bar\psi\gamma^\mu\psi + \frac{1}{2}\bar\psi\gamma^\mu\psi = \bar\psi\gamma^\mu\psi \end{aligned}$$

4. **External symmetries (for scalar fields):**

- *Variation:*

$$\begin{aligned} x^\mu \to x'^\mu &= x^\mu + \delta x^\mu, \quad \phi(x) \to \phi'(x) = \phi(x) + \delta\phi(x), \quad \delta\phi(x) = -\delta x^\mu\partial_\mu\phi(x) \\ \partial_\mu\delta\phi(x) &= \partial_\mu[\phi(x - \delta x^\mu(x)) - \phi(x)] = \partial_\mu\phi(x - \delta x^\mu(x)) - \partial_\mu\phi(x) = \delta\partial_\mu\phi(x) \end{aligned}$$

- *Action:*



$$\begin{aligned}
\delta S &= \int_V dx \delta L + \int_{V'-V} dx L \\
&= \int_V dx \delta L + \int_{\partial V} dS_\mu \delta x^\mu L \\
&= \int_V dx \left(\frac{\partial L}{\partial \phi} - \partial_\mu \frac{\partial L}{\partial \partial_\mu \phi} \right) \delta \phi + \int_{\partial V} dS_\mu \left(\frac{\partial L}{\partial \partial_\mu \phi} \delta \phi + \delta x^\mu L \right) \\
&= \int_{\partial V} dS_\mu \delta x^\nu \left(L g^\mu_\nu - \frac{\partial L}{\partial \partial_\mu \phi} \partial_\nu \phi \right)
\end{aligned}$$

5. **(Canonical) Energy-momentum tensor:**

- *Homogeneous shift:* $\delta x^\mu = \epsilon^\mu$ in arbitrary volume V
- *Conserved current:*

$$\begin{aligned}
J^\mu &= \epsilon^\nu \left(L g^\mu_\nu - \frac{\partial L}{\partial \partial_\mu \phi} \partial_\nu \phi \right) \\
T^{\mu\nu}_c &= \frac{\partial L}{\partial \partial_\mu \phi} \partial^\nu \phi - L g^{\mu\nu} = T^{\nu\mu}_c, \quad \partial_\mu T^{\mu\nu}_c = 0
\end{aligned}$$

- *Energy-momentum:*

$$\begin{aligned}
P^\nu &= \int d^3x T^{0\nu}_c \\
T^{\mu\nu}_c &= \begin{pmatrix} \epsilon & \boldsymbol{p} \\ \boldsymbol{S} & \sigma \end{pmatrix}
\end{aligned}$$

(a) ϵ: energy density

(b) $\boldsymbol{p}$: momentum density

(c) $\boldsymbol{S}$: density of the energy flux

(d) σ^{jk}: flux of p^k in the direction j

6. **Relativistic angular momentum:**

- *Lorentz transformation and translation:* $\delta x^\mu = a^\mu + \omega^\mu_\nu x^\nu$, $\delta\phi = \Lambda^{\nu\mu}\omega_{\mu\nu}\phi \neq 0$

- *Conserved current:*

$$\begin{aligned} J^\mu &= \frac{\partial L}{\partial\partial_\mu\phi}(\Lambda^{\nu\kappa}\omega_{\kappa\nu}\phi - \delta x^\nu\partial_\nu\phi) + \delta x^\mu L \\ &= f^{\mu\nu\kappa}\omega_{\kappa\nu} - \frac{\partial L}{\partial\partial_\mu\phi}\delta x^\nu\partial_\nu\phi + \delta x^\mu L, \quad f^{\mu\nu\kappa} = \frac{\partial L}{\partial\partial_\mu\phi}\Lambda^{\nu\kappa}\phi \end{aligned}$$

cyclic permutation of the indices $\mu\nu\kappa$:

$$\tilde f^{\mu\nu\kappa} = \left(\frac{\partial L}{\partial\partial_\mu\phi}\Lambda^{\nu\kappa} + \frac{\partial L}{\partial\partial_\nu\phi}\Lambda^{\kappa\mu} - \frac{\partial L}{\partial\partial_\kappa\phi}\Lambda^{\mu\nu}\right)\phi$$

antisymmetrical in the first two indices

$$\begin{aligned} \tilde f^{\nu\mu\kappa} &= \left(\frac{\partial L}{\partial\partial_\nu\phi}\Lambda^{\mu\kappa} + \frac{\partial L}{\partial\partial_\mu\phi}\Lambda^{\kappa\nu} - \frac{\partial L}{\partial\partial_\kappa\phi}\Lambda^{\nu\mu}\right)\phi \\ &= \left(-\frac{\partial L}{\partial\partial_\nu\phi}\Lambda^{\kappa\mu} - \frac{\partial L}{\partial\partial_\mu\phi}\Lambda^{\nu\kappa} + \frac{\partial L}{\partial\partial_\kappa\phi}\Lambda^{\mu\nu}\right)\phi \\ &= -\tilde f^{\mu\nu\kappa} \end{aligned}$$

and satisfies

$$\begin{aligned} \tilde f^{\mu\nu\kappa}\omega_{\nu\kappa} &= \left(\frac{\partial L}{\partial\partial_\mu\phi}\Lambda^{\nu\kappa} + \frac{\partial L}{\partial\partial_\nu\phi}\Lambda^{\kappa\mu} - \frac{\partial L}{\partial\partial_\kappa\phi}\Lambda^{\mu\nu}\right)\phi\omega_{\nu\kappa} \\ &= f^{\mu\nu\kappa}\omega_{\nu\kappa} - \left(\frac{\partial L}{\partial\partial_\nu\phi}\Lambda^{\mu\kappa} + \frac{\partial L}{\partial\partial_\kappa\phi}\Lambda^{\mu\nu}\right)\phi\omega_{\nu\kappa} \\ &= f^{\mu\nu\kappa}\omega_{\nu\kappa}. \end{aligned}$$

hence

$$\begin{aligned} J^\mu &= \tilde f^{\mu\nu\kappa}\omega_{\kappa\nu} - \frac{\partial L}{\partial\partial_\mu\phi}\delta x^\nu\partial_\nu\phi + \delta x^\mu L \\ &= \tilde f^{\mu\nu\kappa}\partial_\nu(\delta x_\kappa) - \frac{\partial L}{\partial\partial_\mu\phi}\delta x^\nu\partial_\nu\phi + \delta x^\mu L \\ &= \delta x_\kappa\left(g^{\mu\kappa}L - \frac{\partial L}{\partial\partial_\mu\phi}\partial^\kappa\phi - \partial_\nu\tilde f^{\mu\nu\kappa}\right) + \underbrace{\partial_\nu(\tilde f^{\mu\nu\kappa}\delta x_\kappa\phi)}_{conserved} \end{aligned}$$

- *Modified conserved current:*

$$\begin{aligned} J^\mu &= \delta x_\kappa\left(g^{\mu\kappa}L - \frac{\partial L}{\partial\partial_\mu\phi}\partial^\kappa\phi - \partial_\nu\tilde f^{\mu\nu\kappa}\right) \\ &= T^{\mu\nu}(a_\nu + \omega_{\nu\kappa}x^\kappa) = T^{\mu\nu}a_\nu + \frac{1}{2}(T^{\mu\nu}x^\kappa - T^{\mu\kappa}x^\nu)\omega_{\nu\kappa} \end{aligned}$$

- *Non-canonical energy momentum tensor:*

$$T^{\mu\nu} = T_c^{\mu\nu} + \partial_\kappa\tilde f^{\mu\kappa\nu}$$

(a) same energy-momentum content than $T_c^{\mu\nu}$:

$$P^\nu = \int d^3xT^{0\nu} = \int d^3x(T_c^{0\nu} + \partial_\kappa\tilde f^{0\kappa\nu}) = \int d^3x(T_c^{0\nu} + \partial_j\tilde f^{0j\nu}) = \int d^3xT_c^{0\nu}$$

(b) energy-momentum conservation:

$$\int_{\partial V} dS_\mu \partial_\kappa \tilde{f}^{\mu\kappa\nu} = \int_V \partial_\mu \partial_\kappa \tilde{f}^{\mu\kappa\nu} = 0$$

(c) symmetrical:

$$\begin{aligned} M^{\mu\nu\sigma} &= T^{\mu\nu} x^\sigma - T^{\mu\sigma} x^\nu, \qquad \partial_\mu M^{\mu\nu\sigma} = 0 \\ 0 &= \partial_\rho M^{\rho\mu\nu} = \partial_\rho (T^{\rho\mu} x^\nu - T^{\rho\nu} x^\mu) = T^{\nu\mu} - T^{\mu\nu} \end{aligned}$$

- *Angular momentum:*

$$J^{\nu\sigma} = \int d^3x M^{0\nu\sigma} = \int d^3x (T^{0\nu} x^\sigma - T^{0\sigma} x^\nu).$$

III. CANONICAL QUANTIZATION

A. Single particle

- Classical system:

1. Lagrangian: $L(\boldsymbol{x}, \dot{\boldsymbol{x}})$
2. Canonical momentum:

$$\boldsymbol{p} = \frac{\partial L(\boldsymbol{x}, \dot{\boldsymbol{x}})}{\partial \dot{\boldsymbol{x}}}$$

3. Canonical structure (Poisson brackets):

$$\{x_j, p_k\} = \delta_{j,k}, \qquad \{x_j, x_k\} = \{p_j, p_k\} = 0$$

4. Hamiltonian:

$$H = \dot{\boldsymbol{x}} \boldsymbol{p} - L(\boldsymbol{x}, \dot{\boldsymbol{x}})$$

5. E.O.M.:

$$\dot{f} = \{f, H\}$$

- Quantum system:

1. Lagrangian: $L(\boldsymbol{x}, \dot{\boldsymbol{x}})$
2. Canonical momentum:

$$\boldsymbol{p} = \frac{\partial L(\boldsymbol{x}, \dot{\boldsymbol{x}})}{\partial \dot{\boldsymbol{x}}}$$

3. Canonical structure (commutation relations):

$$[\hat{x}_j, \hat{p}_k] = i\delta_{j,k}, \qquad [\hat{x}_j, \hat{x}_k] = [\hat{p}_j, \hat{p}_k] = 0$$
$$\hat{\boldsymbol{x}}\psi(\boldsymbol{x}) = \boldsymbol{x}\psi(\boldsymbol{x}), \qquad \hat{\boldsymbol{p}}\psi(\boldsymbol{x}) = \frac{1}{i}\frac{\partial}{\partial \boldsymbol{x}}\psi(\boldsymbol{x}).$$

4. Hamiltonian:

$$H = \dot{\boldsymbol{x}}\boldsymbol{p} - L(\boldsymbol{x}, \dot{\boldsymbol{x}}),$$

5. Schrödinger's equation:

$$i\partial_0|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$$

6. Wave functions: $\psi(\boldsymbol{x}) = \langle \boldsymbol{x}|\psi\rangle$, coorindate eigenstates $\hat{\boldsymbol{x}}|\boldsymbol{x}\rangle = \boldsymbol{x}|\boldsymbol{x}\rangle$

B. Quantum field for free particles

1. QM and the continuity of space-time are incompatible $\implies$ $\boldsymbol{x} \to \boldsymbol{x}_n = \Delta x \boldsymbol{n}$, $\psi(t, \boldsymbol{x}) \to \psi(t, \boldsymbol{x}_n)$

2. Quantization box: N^3 lattice sites,

   - Minimal distance, UV cutoff, Δx

   - Maximal distance (IR cutoff) $N\Delta x$

3. Physics in a double limit: $\Delta x \to 0$, $N\Delta x \to \infty$

4. QFT of $\phi_{\boldsymbol{n}} = \phi(\Delta x \boldsymbol{n})$: QM in N^3 dimensions

Field quantization:

1. Action:

$$S = \Delta x^3 \int dt \sum_{\boldsymbol{n}} L(\phi_{\boldsymbol{n}}, \partial_0\phi_{\boldsymbol{n}}, \nabla\phi_{\boldsymbol{n}}) = \int dt L_t[\phi, \partial_0\phi]$$

2. Lagrangian:

$$L_t[\phi, \partial_0\phi] = \Delta x^3 \sum_{\boldsymbol{n}} L(\phi_{\boldsymbol{n}}, \partial_0\phi, \boldsymbol{\nabla}\phi_{\boldsymbol{n}}), \quad \nabla_j\phi(\boldsymbol{x}) = \frac{\phi_{\boldsymbol{n}+\boldsymbol{e}_j} - \phi_{\boldsymbol{n}}}{\Delta x},$$

   and $\boldsymbol{e}_j$ is the unit vector in the direction j ($(\boldsymbol{\nabla}\phi_{\boldsymbol{n}})^2$ is part of potential energy)

3. Functional derivative:

$$\frac{\delta}{\delta\phi(a\boldsymbol{n})} = \frac{1}{\Delta x^3}\frac{\partial}{\partial\phi_{\boldsymbol{n}}}$$
$$\frac{\delta}{\delta\phi(\boldsymbol{x})}\int d^3y f(\boldsymbol{y})\phi(\boldsymbol{y}) = f(\boldsymbol{x}) \quad \Longleftrightarrow \quad \frac{1}{\Delta x^3}\frac{\partial}{\partial\phi_{\boldsymbol{n}}}\Delta x^3 \sum_{\boldsymbol{n}'} f_{\boldsymbol{n}'}\phi_{\boldsymbol{n}'} = f_{\boldsymbol{n}}$$

4. Canonical momentum:

$$\pi_{\boldsymbol{n}} = \frac{\partial L_t[\phi, \nabla\phi]}{\partial\nabla_0\phi_{\boldsymbol{n}}} = \Delta x^3 \frac{\partial L(\phi(x), \partial\phi(x))}{\partial\partial_0\phi(a\boldsymbol{n})}$$

5. Hamiltonian:

$$H = \sum_{\boldsymbol{n}} \partial_0\phi_{\boldsymbol{n}}\pi_{\boldsymbol{n}} - L_t[\phi, \partial_0\phi] \to \int d^3x[\partial_0\phi(\boldsymbol{x})\pi(\boldsymbol{x}) - L(\phi(\boldsymbol{x})), \partial\phi(\boldsymbol{x})))]$$

6. Commutation relations,

$$[\hat{\phi}_{\boldsymbol{n}}, \hat{\pi}_{\boldsymbol{n}'}] = i\delta_{\boldsymbol{n},\boldsymbol{n}'}, \qquad [\hat{\phi}_{\boldsymbol{n}}, \hat{\phi}_{\boldsymbol{n}'}] = [\hat{\pi}_{\boldsymbol{n}}, \hat{\pi}_{\boldsymbol{n}'}] = 0.$$

$\Delta x \to 0$:

$$[\hat{\phi}(\boldsymbol{x}), \hat{\pi}(\boldsymbol{y})] = i\delta(\boldsymbol{x} - \boldsymbol{y}) = i\frac{\delta_{\boldsymbol{n},\boldsymbol{m}}}{\Delta x^3}, \qquad [\hat{\phi}(\boldsymbol{x}), \hat{\phi}(\boldsymbol{y})] = [\hat{\pi}(\boldsymbol{x}), \hat{\pi}(\boldsymbol{y})] = 0$$

7. Wave functionals, canonical coordinate and momentum operators:

$$\begin{aligned}
\phi(\boldsymbol{x})|\Phi(\boldsymbol{y})\rangle &= \Phi(\boldsymbol{x})|\Phi(\boldsymbol{y})\rangle \\
\Psi[\phi(\boldsymbol{y})] &= \langle\phi(\boldsymbol{y})|\Psi\rangle \\
\hat{\phi}(\boldsymbol{x})\Psi[\phi(\boldsymbol{y})] &= \phi(\boldsymbol{x})\Psi[\phi(\boldsymbol{y})] \\
\hat{\pi}(\boldsymbol{x})\Psi[\phi(\boldsymbol{y})] &= \frac{1}{i}\frac{\delta}{\delta\phi(\boldsymbol{x})}\Psi[\phi(\boldsymbol{y})]
\end{aligned}$$

8. Schrödinger's equation:

$$i\partial_0|\Psi\rangle = \hat{H}|\Psi\rangle$$

Example:

$$\begin{aligned}
L &= \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - U(\phi) \\
\pi(\boldsymbol{x}) &= \partial_0\phi(\boldsymbol{x})) \\
H &= \int d^3x \left[-\frac{1}{2}\frac{\delta^2}{\delta\phi(\boldsymbol{x})\delta\phi(\boldsymbol{x})} + \frac{1}{2}(\boldsymbol{\partial}\phi(\boldsymbol{x}))^2 + \frac{m^2}{2}\phi^2(\boldsymbol{x}) + U(\phi(\boldsymbol{x}))\right].
\end{aligned}$$

Too complicated, a shortened construction follows:

- free fields: explicite solution of the equations of motion

- interactions: perturbation expansion

C. Neutral scalar field ($S = 0$)

- General solution of the Klein-Gordon equation:

$$\phi(x) = \int \tilde{d\boldsymbol{k}}[a(\boldsymbol{k})e^{-ikx} + b^\dagger(\boldsymbol{k})e^{ikx}]$$

- Hermiticity:

$$\phi(x) = \int \tilde{d\boldsymbol{k}}[a(\boldsymbol{k})e^{-ikx} + b^\dagger(\boldsymbol{k})e^{ikx}] = \phi^\dagger(x) = \int \tilde{d\boldsymbol{k}}[a^\dagger(\boldsymbol{k})e^{ikx} + b(\boldsymbol{k})e^{-ikx}] \quad \Longrightarrow \quad a(k) = b(k)$$

$$\boxed{\text{Neutral particles and anti-particles are identical}}$$

- Canonical commutation relation:

$$\begin{aligned}
L &= \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - U(\phi) \\
\pi(\boldsymbol{x}) &= \partial_0\phi(\boldsymbol{x})) = -i\int \tilde{dk}\omega_{\boldsymbol{k}}[a(\boldsymbol{k})e^{-ik\cdot x} - a^\dagger(\boldsymbol{k})e^{ik\cdot x}] \\
[\phi(x), \pi(y)]_{|x^0=y^0} &= -i\int \tilde{d\boldsymbol{k}}\tilde{d\boldsymbol{\ell}}\omega_{\boldsymbol{\ell}}[a(\boldsymbol{k})e^{-ikx} + a^\dagger(\boldsymbol{k})e^{ikx}, a(\boldsymbol{\ell})e^{-i\ell y} - a^\dagger(\boldsymbol{\ell})e^{i\ell y}]_{|x^0=y^0} \\
&= -i\int \tilde{d\boldsymbol{k}}\tilde{d\boldsymbol{\ell}}\omega_{\boldsymbol{\ell}}\Big\{\underbrace{[a(\boldsymbol{k}), a(\boldsymbol{\ell})]}_{0}e^{-i(kx+\ell y)} - \underbrace{[a^\dagger(\boldsymbol{k}), a^\dagger(\boldsymbol{\ell})]}_{0}e^{i(kx+\ell y)} \\
&\quad + \underbrace{[a^\dagger(\boldsymbol{k}), a(\boldsymbol{\ell})]}_{-(2\pi)^3 2\omega_{\boldsymbol{k}}\delta(\boldsymbol{k}-\boldsymbol{\ell})} e^{i(kx-\ell y)} - \underbrace{[a(\boldsymbol{k}), a^\dagger(\boldsymbol{\ell})]}_{(2\pi)^3 2\omega_{\boldsymbol{k}}\delta(\boldsymbol{k}-\boldsymbol{\ell})} e^{-i(kx-\ell y)}\Big\}_{|x^0=y^0} \\
&= i\delta(\boldsymbol{x}-\boldsymbol{y}) = i\int \frac{d^3k}{(2\pi)^3}e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} \\
\left[a(\boldsymbol{k}), a^\dagger(\boldsymbol{\ell})\right] &= (2\pi)^3 2\omega_{\boldsymbol{k}}\delta(\boldsymbol{k}-\boldsymbol{\ell}), \quad [a(\boldsymbol{k}), a(\boldsymbol{\ell})] = [a^\dagger(\boldsymbol{k}, a^\dagger(\boldsymbol{\ell})] = 0
\end{aligned}$$

Lesson: $a(\boldsymbol{k})$ and $a^\dagger(\boldsymbol{k})$ are annihilation and creation operators

- Energy-momentum:

$$\begin{aligned}
P^\mu &= \int d^3x T^{0\mu} \\
&= \int d^3x \left[\partial_0\phi\partial^\mu\phi - g^{0\mu}\left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2\right)\right] \\
&= \frac{1}{2}\int \tilde{d\boldsymbol{k}}k^\mu[\underbrace{a(\boldsymbol{k})a^\dagger(\boldsymbol{k})}_{a(\boldsymbol{k})a^\dagger(\boldsymbol{k})+(2\pi)^2 2\omega_{\boldsymbol{k}}\delta(\boldsymbol{0})} + a^\dagger(\boldsymbol{k})a(\boldsymbol{k})]
\end{aligned}$$

Dirac-delta at zero? $x_j = j\Delta x$, $\Delta k = \frac{2\pi}{L}$, $\frac{1}{\Delta k^3} = \frac{V}{(2\pi)^3}$

$$\int dx\delta(x - x_0)f(x_0) = \Delta x\sum_j \underbrace{"\delta(x_j)"}_{\frac{1}{\Delta x}\delta_{j,j_0}} f(x_j) = f(x_{j_0})$$

$$P^\mu = \int \tilde{dk} k^\mu a^\dagger(\boldsymbol{k})a(\boldsymbol{k}) + \frac{V}{2}\int \frac{d^3k}{(2\pi)^3}k^\mu\omega_{\boldsymbol{k}}$$
$$= \int \tilde{dk} k^\mu a^\dagger(\boldsymbol{k})a(\boldsymbol{k}) + g^{0\mu}V\int \frac{d^3k}{(2\pi)^3}\frac{\omega_{\boldsymbol{k}}}{2}$$

Diverging vacuum energy density:

1. UV divergence: Sum of the ground state energy of infinitely many harmonic oscillator
2. Naive removal: normal ordering, creation (destruction) operators left (right),

   $$: a(\boldsymbol{p}_1)a(\boldsymbol{p}_1) := a(\boldsymbol{p}_2)a(\boldsymbol{p}_1),, \qquad : a(\boldsymbol{p}_1)a^\dagger(\boldsymbol{p}_2) := a^\dagger(\boldsymbol{p}_2)a(\boldsymbol{p}_1)$$

   $\langle 0| : A : |\rangle = 0$, $P^\mu \to: P^\mu :$
3. Regulator: Maximal momentum, $|\boldsymbol{p}| < \Lambda$, our ignorance of microscopic laws
4. Renormalization: $\frac{1}{\Lambda} \ll$ observed length scale, the renormalized theory is constructed by the limit $\Lambda \to \infty$. Convergent theories are called renormalizable.

D. Charged scalar field ($S = 0$)

1. Syllogism: Conservation law $\implies$ symmetry
2. Single charge $\implies$ one dimensional continuous symmetry group, $G = U(1)$
3. $\phi(x) = \frac{1}{\sqrt{2}}[\phi_1(x) + i\phi_2(x)]$, $\phi(x) \to e^{i\alpha}\phi(x)$
4. Lagrangian:

   $$L_{free} = \sum_{a=1,2}\left[\frac{1}{2}\partial_\mu\phi_a\partial^\mu\phi_a - \frac{m^2}{2}\phi_a^2\right] = \partial_\mu\phi^\dagger\partial^\mu\phi - m^2\phi^\dagger\phi$$
   $$L_{int} = L_{free} - U(\phi^\dagger\phi)$$
5. Canonical momentum:

   $$\pi(x) = \frac{\partial L}{\partial\partial_0\phi} = \partial_0\phi^\dagger, \qquad \pi^\dagger(x) = \frac{\partial L}{\partial\partial_0\phi^\dagger} = \partial_0\phi = (\pi(x))^\dagger$$
6. Canonical commutation relations:

   $$i\delta(\boldsymbol{x}-\boldsymbol{y}) = [\phi(\boldsymbol{x}), \pi(\boldsymbol{y})] = [\phi^\dagger(\boldsymbol{x}), \pi^\dagger(\boldsymbol{y})]$$
   $$0 = [\phi(\boldsymbol{x}), \phi(\boldsymbol{y})] = [\phi(\boldsymbol{x}), \phi^\dagger(\boldsymbol{y})] = [\pi(\boldsymbol{x}), \pi(\boldsymbol{y})] = [\pi(\boldsymbol{x}), \pi^\dagger(\boldsymbol{y})]$$

7. Quantum field:

$$\phi(t,\boldsymbol{x}) = \int \tilde{d\boldsymbol{k}}[a(\boldsymbol{k})e^{-ik\cdot x} + b^\dagger(\boldsymbol{k})e^{ik\cdot x}]$$

8. Canonical commutation relations:

$$\begin{aligned}(2\pi)^3 2\omega_{\boldsymbol{k}}\delta(\boldsymbol{k}-\boldsymbol{\ell}) &= [a(\boldsymbol{k}),a^\dagger(\boldsymbol{\ell})] = [b(\boldsymbol{k}),b^\dagger(\boldsymbol{\ell})]\\ 0 &= [a(\boldsymbol{k}),a(\boldsymbol{\ell})] = [a^\dagger(\boldsymbol{k}),a^\dagger(\boldsymbol{\ell})] = [a(\boldsymbol{k}),b(\boldsymbol{\ell})] = [a(\boldsymbol{k}),b^\dagger(\boldsymbol{\ell})]\end{aligned}$$

Two kind of particles (particle and anti-particle), identical dispersion relation, $\omega_{\boldsymbol{k}}$.

9. Energy-momentum:

$$\begin{aligned}P^\mu &= \int d^3x T^{0\mu}\\ &= \int d^3x : \left[\partial_0\phi^\dagger\partial^\mu\phi + \partial^\mu\phi^\dagger\partial_0\phi - g^{0\mu}(\partial_\mu\phi^\dagger\partial^\mu\phi - m^2\phi^\dagger\phi)\right] :\\ &= \frac{1}{2}\int \tilde{d\boldsymbol{k}}k^\mu : [a(\boldsymbol{k})a^\dagger(\boldsymbol{k}) + a^\dagger(\boldsymbol{k})a(\boldsymbol{k}) + b(\boldsymbol{k})b^\dagger(\boldsymbol{k}) + b^\dagger(\boldsymbol{k})b(\boldsymbol{k})] :\\ &= \int \tilde{d\boldsymbol{k}}k^\mu[a^\dagger(\boldsymbol{k})a(\boldsymbol{k}) + b^\dagger(\boldsymbol{k})b(\boldsymbol{k})].\end{aligned}$$

10. Charge:

$$Q(t) = \int d^3x j^0(t,\boldsymbol{x}) \int \tilde{d\boldsymbol{k}}[a^\dagger(\boldsymbol{k})a(\boldsymbol{k}) - b^\dagger(\boldsymbol{k})b(\boldsymbol{k})]$$

Noether current

$$j_\mu = i : \phi^\dagger \overleftrightarrow{\partial}_\mu \phi :$$

is the electromagnetic current, $f(x)\overleftrightarrow{\partial}_\mu g(x) = f(x)\partial_\mu g(x) - (\partial_\mu f(x))g(x)$.

E. Neutral vector field ($S = 1$)

1. $S = 1 \implies$ 3 components, relativity ? (scalar 0, vector 4, $\neq 3$)

2. Gauge theories:

   - vector field $A_\mu(x)$ with local relativistic invariant constraint
   - $4 \to 3$: one constraint, one dimensional symmetry (gauge) group,

$$\begin{aligned}A_\mu(x) &\to A_\mu(x) + \partial_\mu\Phi(x)\\ \int dx e^{iqx}A_\mu(x) = A_\mu(q) &\to A_\mu(q) + \int dx e^{iqx}\partial_\mu\Phi(x) = A_\mu(q) - iq_\mu\Phi(q)\end{aligned}$$

3. Maxwell Lagrangian:

$$L = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

4. Mass, a term like $-m^2A^2$, is excluded by gauge invariance

5. Singular Lagrangian:

   - $\partial_0 A_0$ has no dynamics,
   - $\Pi^0 = 0$,

$$\begin{aligned}\Pi^\mu &= \frac{\delta L}{\delta\partial_0 A_\mu} \\ &= -\frac{1}{4}\frac{\delta}{\delta\partial_0 A_\mu}(\partial_\mu A_\nu - \partial_\nu A_\mu)g^{\mu\mu'}g^{\nu\nu'}(\partial_{\mu'}A_{\nu'} - \partial_{\nu'}A_{\mu'}) \\ &= -\frac{1}{2}(\partial^0 A^\mu - \partial^\mu A^0) + \frac{1}{2}(\partial^\mu A_0 - \partial^0 A^\mu) \\ &= F^{\mu 0}\end{aligned}$$

   - Further reduction of the number of degrees of freedom: $4 \to 3 \to 2$
     - $\mathcal{O}\left(m^2\right)$ coupling between longitudinal and transverse helicity components
     - no longitudinal photons

6. Observables: remain invariant under gauge transformations

   - Gauge non-invariant sector:
     - auxiliary (redundant parametrization induced) dynamics
     - freely modifiable
     - longitudinal field components ($\neq$ helicity components!)

$$A_\mu(q) \to A_\mu(q) - iq_\mu\Phi(x), \quad A_{L\mu}(x) = L_{\mu\nu}A^\nu(x), \quad L_{\mu\nu}(k) = \frac{k_\mu k_\nu}{k^2}, \quad L_{\mu\nu} = \frac{\partial_\mu\partial_\nu}{\Box}$$

   - Gauge invariant sector: physics, transverse field components

$$A_{T\mu}(x) = T_{\mu\nu}A^\nu(x), \quad T_{\mu\nu}(k) = g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}, \quad T_{\mu\nu} = g_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box}$$

7. Gauge fixing:

   - non-singular Lagrangian
   - well defined time evolution in gauge theories with symmetry $A_\mu(x) \to A_\mu(x) + \partial_\mu\Phi(x)$

- modification of the gauge-dependent sector

$$L = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{\lambda}{2}(\partial^\mu A_\mu)^2$$

- E.O.M.:

$$\begin{aligned}
0 &= \left(\frac{\delta}{\delta A_\mu} - \partial_\nu \frac{\delta}{\delta \partial_\nu A_\mu}\right)\left[-\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)g^{\mu\mu'}g^{\nu\nu'}(\partial_{\mu'}A_{\nu'} - \partial_{\nu'}A_{\mu'}) - \frac{\lambda}{2}(\partial^\mu A_\mu)^2\right] \\
&= -\partial_\nu\left(-\frac{1}{4}\right)(F^{\nu\mu} - F^{\mu\nu} + F^{\nu\mu} - F^{\mu\nu}) + \lambda g^{\mu\nu}\partial_\nu\partial_\mu A_\mu \\
&= \partial_\nu F^{\nu\mu} + \lambda g^{\mu\nu}\partial_\nu\partial_\mu A_\mu \\
&= \partial_\nu(\partial^\nu A^\mu - \partial^\mu A^\nu) + \lambda g^{\mu\nu}\partial_\nu\partial_\mu A_\mu \\
&= \Box A^\mu + (\lambda - 1)\partial^\mu\partial_\nu A^\nu
\end{aligned}$$

- generalized momentum:

$$\begin{aligned}
\Pi^\mu &= \frac{\delta L}{\delta \partial_0 A_\mu} \\
&= \frac{\delta}{\delta \partial_0 A_\mu}\left[-\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)g^{\mu\mu'}g^{\nu\nu'}(\partial_{\mu'}A_{\nu'} - \partial_{\nu'}A_{\mu'}) - \frac{\lambda}{2}(\partial^\mu A_\mu)^2\right] \\
&= -\frac{1}{2}(\partial^0 A^\mu - \partial^\mu A^0) + \frac{1}{2}(\partial^\mu A_0 - \partial^0 A^\mu) - \lambda g^{\mu 0}\partial^\nu A_\nu \\
&= F^{\mu 0} - \lambda g^{\mu 0}\partial^\nu A_\nu
\end{aligned}$$

8. Canonical commutation relations:

- relativistic covariance:

$$[A_\mu(t, \boldsymbol{x}), \pi_\nu(t, \boldsymbol{y})] = \pm i g_{\mu\nu}\delta(\boldsymbol{x} - \boldsymbol{y})$$

- to get most of the components right:

$$[A_\mu(t, \boldsymbol{x}), \pi_\nu(t, \boldsymbol{y})] = -i g_{\mu\nu}\delta(\boldsymbol{x} - \boldsymbol{y})$$

- A_0 has commutation relation with the wrong sign
- Fock-space with indefinite scalar product

$$\begin{aligned}
[x, p] &= \pm i\hbar, \quad a = \frac{m\omega x + ip}{\sqrt{2\hbar\omega m}} \\
\left[a, a^\dagger\right] &= \left[\frac{m\omega x + ip}{\sqrt{2\hbar\omega m}}, \frac{m\omega x - ip}{\sqrt{2\hbar\omega m}}\right] = \pm 1 \\
\langle 0|0\rangle &= 1 \\
\langle 1|1\rangle &= \langle 0|a a^\dagger|0\rangle = \langle 0|a^\dagger a \pm 1|0\rangle = \pm\langle 0|0\rangle = \pm 1
\end{aligned}$$

- Gupta-Bleuer quantization: positive definite states in the physical (gauge invariant) subspace

9. Quantum field:

$$A_\mu(x) = \int d\tilde{\boldsymbol{k}} \sum_{\lambda=0}^{3} [a_\lambda(\boldsymbol{k})\epsilon_{\lambda\mu}(\boldsymbol{k})e^{-ik\cdot x} + a_\lambda^\dagger(\boldsymbol{k})\epsilon_{\lambda\mu}(\boldsymbol{k})e^{ik\cdot x}],$$

polarization vectors $\epsilon_{\lambda\mu}(\boldsymbol{k})$:

- time-like polarization: $\epsilon_{0\mu}(\boldsymbol{k}) = n^\mu$, $n^\mu n_\mu = 1$, time-like unit vector
- longitudinal polarization $\epsilon_{3\mu}(\boldsymbol{k}) = (\omega_{\boldsymbol{k}}, \boldsymbol{k})$
- transverse polarization: $\epsilon_{\lambda\mu}(\boldsymbol{k})$, $\lambda = 1, 2$, are orthogonal to $\epsilon_{0\mu}(\boldsymbol{k})$ and $\epsilon_{3\mu}(\boldsymbol{k})$

$$\epsilon_\lambda^\mu(\boldsymbol{k})\epsilon_{\lambda'\mu}^*(\boldsymbol{k}) = g_{\lambda\lambda'}.$$

- non-vanishing canonical commutation relations:

$$[a_\lambda(\boldsymbol{k}), a_{\lambda'}^\dagger(\boldsymbol{\ell})] = -g_{\lambda\lambda'} 2\omega_{\boldsymbol{k}}\delta(\boldsymbol{k} - \boldsymbol{\ell})$$

F. Charged fermionic field ($S = \frac{1}{2}$)

1. Dirac equation:

$$(i\partial_\mu\gamma^\mu - m)\psi(x) = (i\partial\!\!\!/ - m)\psi(x) = 0$$

2. Plane wave solution:

$$\begin{aligned} \text{particle} \ &: \quad \psi^{(+)}(x) = e^{-ikx}u_k, \quad (k\!\!\!/ - m)u_k = 0, \quad k^0 > 0 \\ \text{anti}-\text{particle} \ &: \quad \psi^{(-)}(x) = e^{ikx}v_k, \quad (k\!\!\!/ + m)v_k = 0, \quad k^0 > 0 \end{aligned}$$

3. Lagrangian:

$$L = \to \frac{i}{2}[\bar{\psi}\gamma^\mu(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma^\mu\psi] - m\bar{\psi} \to \bar{\psi}[i\partial_\mu\gamma^\mu - m]\psi.$$

4. Energy-momentum tensor: (by the help of E.O.M.)

$$T^{\mu\nu} = \partial^\nu\bar{\psi}\frac{\partial L}{\partial\partial_\mu\bar{\psi}} + \frac{\partial L}{\partial\partial_\mu\psi}\partial^\nu\psi - g^{\mu\nu}L = \frac{i}{2}[\bar{\psi}\gamma^\mu\partial^\nu\psi - \partial^\nu\bar{\psi}\gamma^\mu\psi]$$

5. Noether current: $G = U(1)$, $\psi(x) \to e^{i\alpha}\psi(x)$,

$$j^\mu = \bar{\psi}\gamma^\mu\psi$$

6. Quantum field: $\alpha = 1, 2$ spin polarization,

$$\psi(x) = \int \frac{d^3k}{(2\pi)^3}\frac{m}{\omega_{\boldsymbol{k}}} \sum_{\alpha=1,2} [c_\alpha(\boldsymbol{k})u^{(\alpha)}(\boldsymbol{k})e^{-ik\cdot x} + d_\alpha^\dagger(\boldsymbol{k})v^{(\alpha)}(\boldsymbol{k})(\boldsymbol{k})e^{ik\cdot x}]$$

7. Energy-momentum vector:

$$\begin{aligned} P^\mu &= \int d^3x T^{0\mu} \\ &= \int \frac{d^3k}{(2\pi)^3}\frac{m}{\omega_{\boldsymbol{k}}} k^\mu \sum_{\alpha=1,2} [c^\dagger_\alpha(\boldsymbol{k})c_\alpha(\boldsymbol{k}) - d_\alpha(\boldsymbol{k})d^\dagger_\alpha(\boldsymbol{k})] \end{aligned}$$

the negative sign is the usual wrong sign in relativistic quantum mechanics and is here harmless

8. Canonical anti-commutation relations:

- P^0 bounded from below
- right exchange statisics

$$\begin{aligned} (2\pi)^3 2\omega_{\boldsymbol{k}}\delta_{\alpha,\beta}\delta(\boldsymbol{k}-\boldsymbol{\ell}) &= \{c_\alpha(\boldsymbol{k}), c^\dagger_\beta(\boldsymbol{\ell})\} = \{d_\alpha(\boldsymbol{k}), d^\dagger_\beta(\boldsymbol{\ell})\}, \\ 0 &= \{c_\alpha(\boldsymbol{k}), c_\beta(\boldsymbol{\ell})\} = \{d_\alpha(\boldsymbol{k}), d_\beta(\boldsymbol{\ell})\} = \{c_\alpha(\boldsymbol{k}), d_\beta(\boldsymbol{\ell})\} = \{c_\alpha(\boldsymbol{k}), d^\dagger_\beta(\boldsymbol{\ell})\} \end{aligned}$$

9. Energy-momentum vector after the subtraction of the divergent contribution of the zero-point fluctuations:

$$P^\mu = \int \frac{d^3k}{(2\pi)^3}\frac{m}{\omega_{\boldsymbol{k}}} k^\mu \sum_{\alpha=1,2} [c^\dagger_\alpha(\boldsymbol{k})c_\alpha(\boldsymbol{k}) + d^\dagger_\alpha(\boldsymbol{k})d_\alpha(\boldsymbol{k})].$$

normal ordering: same as for bosonic operators except an exhange sign factor,

e.g. $: c_\alpha(\boldsymbol{p}_1)c_\beta(\boldsymbol{p}_1) := -c_\beta(\boldsymbol{p}_2)c_\alpha(\boldsymbol{p}_1)$, $: c_\alpha(\boldsymbol{p}_1)c^\dagger_\beta(\boldsymbol{p}_2) := -c^\dagger_\beta(\boldsymbol{p}_2)c_\alpha(\boldsymbol{p}_1)$.

10. Noether current (electric current):

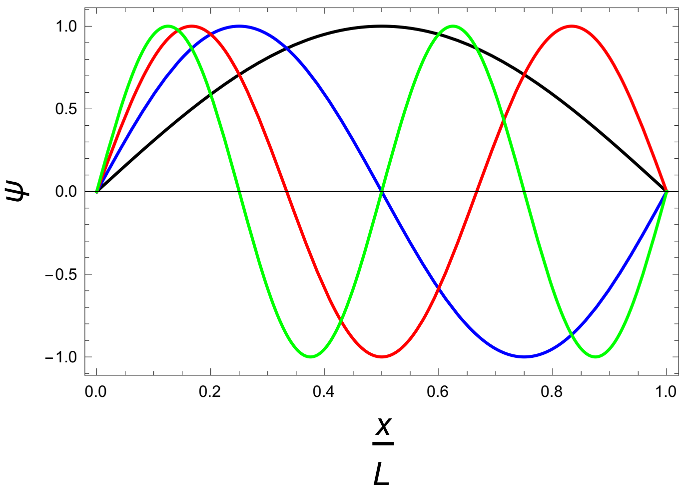
$$Q = \int \frac{d^3k}{(2\pi)^3}\frac{m}{\omega_{\boldsymbol{k}}} \sum_{\alpha=1,2} [c^\dagger_\alpha(\boldsymbol{k})c_\alpha(\boldsymbol{k}) - d^\dagger_\alpha(\boldsymbol{k})d_\alpha(\boldsymbol{k})].$$

IV. SCATTERING PROCESSES

1. Non-relativistic quantum mechanics: $\langle\psi|F(\boldsymbol{x},\boldsymbol{p})|\psi\rangle$
2. Relativistic quantum mechanics: neither $\langle\psi|\boldsymbol{x}|\psi\rangle$ nor $\langle\psi|\boldsymbol{p}|\psi\rangle$ is measurable with arbitrary precision in high energy processes

- $\langle\psi|\boldsymbol{x}|\psi\rangle$: vacuum-polarization pairs at $|\boldsymbol{x}| \ll \lambda_C = \frac{\hbar}{mc}$
  - unidimensional particle confined in $0 < x < L$:

$$\psi_n(x) = \sin k_n x, \quad k_n = n\frac{\pi}{L}, \quad p^2 \sim \frac{\hbar^2}{L^2}$$



- localization needs energy
- maximal localization at the Compton wavelength:

$$E = c\sqrt{m^2c^2 + \boldsymbol{p}^2} \sim c\sqrt{m^2c^2 + \frac{\hbar^2}{L^2}} \sim 3mc^2, \quad \to m^2c^2 \approx \frac{\hbar^2}{L^2}, \quad L \approx \frac{\hbar}{mc} = \lambda_C$$

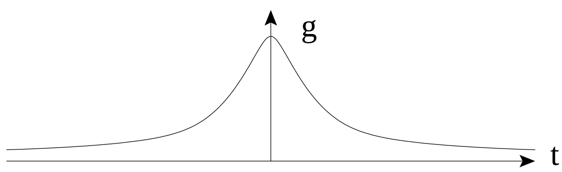
- $\langle\psi|\boldsymbol{p}|\psi\rangle$:
  - high energy: $E = c\sqrt{m^2c^2 + \boldsymbol{p}^2} \sim c|\boldsymbol{p}|$
  - frequency(energy)-time uncertainty relation: $\Delta\omega t \approx 1$, $t \to 0$

$$\Delta E t = \hbar\Delta\omega t \approx \hbar, \quad c\Delta p t \approx \hbar \quad \to \quad \Delta p \approx \frac{\hbar}{ct} \to \infty$$

3. Heisenberg: scattering probabilities is measurable with arbitrary precision

A. Asymptotic states

1. Scattering at high energy:
   - interaction for short time



   - initial and final states, $|i\rangle \in \mathcal{H}_i$ and $|f\rangle \in \mathcal{H}_f$, made of free particles
   - short time interaction $\implies$ well separated wave-packets $\implies$ not too clean beam
2. Scattering matrix in the interaction representation:

$$|i\rangle \to |f\rangle = S|i\rangle, \quad S = T[e^{-i\int_{-\infty}^{\infty} dt H_i(t)}], \quad \mathcal{H}_f \subset S\mathcal{H}_i \ \ \text{(bound states)}$$

3. Asymptotic fields: $\phi_i(x)$ $(x^0 \to -\infty)$, $\phi_o(x)$ $(x^0 \to \infty)$, $(\Box + m^2)\phi_{\substack{i\\o}} = 0$

4. Interpolating field: $\sqrt{Z_{\substack{o\\i}}}\phi(x) = \phi_{\substack{o\\i}}(t, \boldsymbol{x})$ for $x^0 \to \pm\infty$, T-invariance $\implies$ $Z_i = Z_o = Z$

5. $\sqrt{Z}\phi(t, \boldsymbol{x}) = \phi_{\substack{o\\i}}(t, \boldsymbol{x})$ is a weak equation for non-overlapping particles

   - weak equations hold for matrix elements only, $\langle\psi|\sqrt{Z}\phi(t, \boldsymbol{x})|\psi'\rangle = \langle\psi|\phi_{\substack{o\\i}}(t, \boldsymbol{x})|\psi'\rangle$
   - two-point functions

$$\begin{aligned}
\langle 0|\phi(x)\phi(y)|0\rangle &= \sum_n \langle 0|\phi(x)|n\rangle\langle n|\phi(y)|0\rangle \\
\langle 0|\phi_{\substack{o\\i}}(x)\phi_{\substack{o\\i}}(y)|0\rangle &= \sum_n \langle 0|\phi_{\substack{o\\i}}(x)|n\rangle\langle n|\phi_{\substack{o\\i}}(y)|0\rangle
\end{aligned}$$

   - $\boldsymbol{x} \sim \boldsymbol{y}$: particles overlap and

$$Z\langle 0|\phi(x)\phi(y)|0\rangle \neq \langle 0|\phi_{\substack{o\\i}}(x)\phi_{\substack{o\\i}}(y)|0\rangle$$

B. Cross section

1. Inelastic $p_1, p_2 \to q_1, q_2, \ldots, q_n$ scattering:

$$|i\rangle = \int \tilde{d}\boldsymbol{p}_1 \tilde{d}\boldsymbol{p}_2 \psi_1(\boldsymbol{p}_1)\psi_2(\boldsymbol{p}_1)|\boldsymbol{p}_1, \boldsymbol{p}_2\rangle_i, \quad |f\rangle = |\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots, \boldsymbol{q}_n\rangle_f$$

2. Transition probability: $W_{f\leftarrow i} = |\langle f|i(t = \infty)\rangle|^2$

3. Scattering amplitude: $\langle f|i(t = \infty)\rangle = \langle f|S|i\rangle$

4. Ignoring the trivial forward scattering: $S = \mathbb{1} + iT$, $\langle f|S|i\rangle = i\langle f|T|i\rangle$

5. Transition probability:

   - Separation of the trivial energy-momentum conserving Dirac-delta:

$$\begin{aligned}
\langle f|S|i\rangle &= i\langle f|T|i\rangle = \int \tilde{d}\boldsymbol{p}_1 \tilde{d}\boldsymbol{p}_2 \psi_1(\boldsymbol{p}_1)\psi_2(\boldsymbol{p}_1) i(2\pi)^4\delta(p_f - p_1 - p_2)\langle f|\mathcal{T}|\boldsymbol{p}_1, \boldsymbol{p}_2\rangle \\
W_{f\leftarrow i} &= \int \tilde{d}\boldsymbol{p}_1 \tilde{d}\boldsymbol{p}_2 \tilde{d}\boldsymbol{q}_1 \tilde{d}\boldsymbol{q}_2 \psi_1^*(\boldsymbol{p}_1)\psi_2^*(\boldsymbol{p}_1)\psi_1(\boldsymbol{q}_1)\psi_2(\boldsymbol{q}_1)(2\pi)^4\delta(p_f - p_1 - p_2)(2\pi)^4\delta(q_1 + q_2 - p_1 - p_2) \\
&\quad \times\langle f|\mathcal{T}|\boldsymbol{p}_1, \boldsymbol{p}_2\rangle^*\langle f|\mathcal{T}|\boldsymbol{q}_1, \boldsymbol{q}_2\rangle.
\end{aligned}$$

   - Sufficiently monochromatic, but still non-overlapping wave-packets:

$$\langle f|\mathcal{T}|\boldsymbol{p}_1, \boldsymbol{p}_2\rangle \approx \langle f|\mathcal{T}|\bar{\boldsymbol{p}}_1, \bar{\boldsymbol{p}}_2\rangle$$

   ($\bar{p}_j$ is the average momentum of the j-th particle)

6. Transition probability density:

- Initial state wave functions:

$$
\begin{aligned}
f(x) &= \int \tilde{d\boldsymbol{p}} e^{-ixp} f(p)_{|p^0=\omega_{\boldsymbol{q}}} \\
W_{f\leftarrow i} &= \langle f|\mathcal{T}|\bar{\boldsymbol{p}}_1,\bar{\boldsymbol{p}}_2\rangle^* \langle f|\mathcal{T}|\bar{\boldsymbol{p}}_1,\bar{\boldsymbol{p}}_2\rangle \int \tilde{d\boldsymbol{p}_1}\tilde{d\boldsymbol{p}_2}\tilde{d\boldsymbol{q}_1}\tilde{d\boldsymbol{q}_2}\psi_1^*(\boldsymbol{p}_1)\psi_2^*(\boldsymbol{p}_1)\psi_1(\boldsymbol{q}_1)\psi_2(\boldsymbol{q}_1) \\
&\quad \times (2\pi)^4\delta(p_f-p_1-p_2)(2\pi)^4\delta(q_1+q_2-p_1-p_2) \\
&= \langle f|\mathcal{T}|\bar{\boldsymbol{p}}_1,\bar{\boldsymbol{p}}_2\rangle^* \langle f|\mathcal{T}|\bar{\boldsymbol{p}}_1,\bar{\boldsymbol{p}}_2\rangle \int d^4x \tilde{d\boldsymbol{p}_1}\tilde{d\boldsymbol{p}_2}\tilde{d\boldsymbol{q}_1}\tilde{d\boldsymbol{q}_2}\psi_1^*(\boldsymbol{p}_1)\psi_2^*(\boldsymbol{p}_1)\psi_1(\boldsymbol{q}_1)\psi_2(\boldsymbol{q}_1) \\
&\quad \times (2\pi)^4\delta(p_f-p_1-p_2)e^{-ix\cdot(q_1+q_2-p_1-p_2)} \\
&= |\langle f|\mathcal{T}|\bar{\boldsymbol{p}}_1,\bar{\boldsymbol{p}}_2\rangle|^2(2\pi)^4\delta(p_f-\bar{p}_1-\bar{p}_2)\int d^4x|\psi_1(\boldsymbol{x})|^2|\psi_2(\boldsymbol{x})|^2 \\
\frac{dW_{f\leftarrow i}}{dVdt} &= (2\pi)^4\delta(p_f-\bar{p}_1-\bar{p}_2)|\psi_1(\boldsymbol{x})|^2|\psi_2(\boldsymbol{x})|^2|\langle f|\mathcal{T}|\bar{\boldsymbol{p}}_1,\bar{\boldsymbol{p}}_2\rangle|^2
\end{aligned}
$$

- Probability from a unit space-time volume:

7. Factorization of the characteristic quantities of the beam and the target:

- particle 1: beam; particle 2: target in the laboratory rest frame
- target density:

$$
\begin{aligned}
i\psi^*(x)\overleftrightarrow{\partial}_\mu\psi(x) &\approx i(e^{i\bar{p}_2x}\partial_\mu e^{-i\bar{p}_2x}-\partial_\mu e^{i\bar{p}_2x}e^{-i\bar{p}_2x}) = 2\bar{p}_{2,\mu}|\psi(x)|^2 \\
\frac{dn_2}{dV} &= 2\bar{p}_2^0|\psi_2(x)|^2, \qquad \bar{p}_2^0 = m_2
\end{aligned}
$$

- incident beam flux:

$$
\boldsymbol{j}_b = \underbrace{\frac{\bar{\boldsymbol{p}}_1}{\bar{p}_1^0}}_{\text{velocity}} \cdot \underbrace{2\bar{p}_1^0|\psi_1(x)|^2}_{\text{density}} = 2\bar{\boldsymbol{p}}_1|\psi_1(x)|^2
$$

- Lorentz invariant differential cross section:

$$
\begin{aligned}
\frac{dW_{f\leftarrow i}}{dVdt} &= \frac{dn_2}{dV}|\boldsymbol{j}_b|d\sigma \\
d\sigma &= \frac{dW_{f\leftarrow i}}{dVdt}\frac{dV}{dn_2}\frac{1}{|\boldsymbol{j}_b|} \\
&= \frac{(2\pi)^4\delta(p_f-\bar{p}_1-\bar{p}_2)|\psi_1(\boldsymbol{x})|^2|\psi_2(\boldsymbol{x})|^2|\langle f|\mathcal{T}|\bar{\boldsymbol{p}}_1,\bar{\boldsymbol{p}}_2\rangle|^2}{2\bar{p}_2^0|\psi_2(x)|^2 2\bar{\boldsymbol{p}}_1|\psi_1(x)|^2} \\
&= (2\pi)^4\delta(p_f-\bar{p}_1-\bar{p}_2)\frac{|\langle f|\mathcal{T}|\bar{\boldsymbol{p}}_1,\bar{\boldsymbol{p}}_2\rangle|^2}{4m_2|\bar{\boldsymbol{p}}_1|} \\
&= (2\pi)^4\delta(p_f-\bar{p}_1-\bar{p}_2)\frac{|\langle f|\mathcal{T}|\bar{\boldsymbol{p}}_1,\bar{\boldsymbol{p}}_2\rangle|^2}{4\sqrt{(p_1p_2)^2-m_1^2m_2^2}}
\end{aligned}
$$

$$
\left((p_1p_2)^2-m_1^2m_2^2 \to (p_1^0m_2)^2-m_1^2m_2^2 = m_2^2(m_1^2+\boldsymbol{p}_1^2-m_1^2) = m_2^2\boldsymbol{p}_1^2\right)
$$

- Final state momenta constrained in $\Delta \subset \mathbb{R}^{3n}$:

$$d\sigma = \underbrace{\frac{1}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \int_\Delta \tilde{d\boldsymbol{q}_1} \cdots \tilde{d\boldsymbol{q}_n}}_{\text{kinematics}} \overbrace{(2\pi)^4 \delta(p_f - \bar{p}_1 - \bar{p}_2)}^{\text{symmetry}} \underbrace{|\langle \boldsymbol{q}_1 \cdots \boldsymbol{q}_n | \mathcal{T} | \bar{\boldsymbol{p}}_1, \bar{\boldsymbol{p}}_2 \rangle|^2}_{\text{perturbation expansion}}$$

V. PERTURBATION EXPANSION

$$H = H_0 + H_i$$

Example:

$$\begin{aligned}
L &= \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2}\phi^2 - \frac{g}{4!}\phi^4 \\
T_c^{\mu\nu} &= \frac{\partial L}{\partial \partial_\mu \phi}\partial^\nu \phi - L g^{\mu\nu} \\
P^\nu &= \int d^3x T_c^{0\nu} = \int d^3x \left(\frac{\partial L}{\partial \partial_0 \phi}\partial^\nu \phi - L g^{0\nu} \right) \\
H &= \int d^3x \left(\frac{\partial L}{\partial \partial_0 \phi}\partial_0 \phi - L \right) = \int d^3x \left[\frac{1}{2}(\partial_0 \phi)^2 + \frac{1}{2}(\boldsymbol{\partial}\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4 \right] \\
H_0 &= \int d^3x \left[\frac{1}{2}(\partial_0 \phi)^2 + \frac{1}{2}(\boldsymbol{\partial}\phi)^2 + \frac{m^2}{2}\phi^2 \right], \quad H_i = \frac{g}{4!}\int d^3x \phi^4.
\end{aligned}$$

Transition amplitude (in terms of free field operators):

$$\begin{aligned}
|i\rangle &= \phi_i(y_1) \cdots \phi_i(y_{m_i})|0\rangle, \quad |f\rangle = \phi_f(z_1) \cdots \phi_f(z_{m_f})|0\rangle \\
\mathcal{A} &= \langle 0|\phi_f(z_1) \cdots \phi_f(z_{m_f}) T[e^{-i\int dt H_i(t)}]\phi_i(y_1) \cdots \phi_i(y_{m_i})|0\rangle \\
&= Z^{-\frac{1}{2}(n_i+n_f)}\langle 0|T[\phi(z_1) \cdots \phi(z_{m_f}) e^{-i\int dt H_i(t)}\phi(y_1) \cdots \phi(y_{m_i})]|0\rangle \\
&= \langle 0|T[\phi_f(z_1) \cdots \phi_f(z_{m_f}) e^{-\frac{ig}{4!}\int d^4x \phi^4(x)}\phi_f(y_1) \cdots \phi_f(y_{m_i})]|0\rangle \quad \leftarrow Z = 1 \\
&= \sum_{n=0}^\infty \frac{1}{n!}\left(\frac{-ig}{4!}\right)^n \int dx_1 \cdots dx_n \langle 0|T[\phi_f(z_1) \cdots \phi_f(z_{m_f})\phi^4(x_1) \cdots \phi^4(x_n)\phi_f(y_1) \cdots \phi_f(y_{m_i})]|0\rangle
\end{aligned}$$

A. Green functions

Hermitean field:

$$i^{\frac{n}{2}} G(x_1, \ldots, x_n) = \langle 0|T[\phi(x_1) \cdots \phi(x_n)]|0\rangle$$

Non-Hermitean field:

$$i^n G(x_1, \ldots, x_n, y_1, \ldots, y_m) = \langle 0|T[\phi(x_1) \cdots \phi(x_m)\phi^\dagger(y_1) \cdots \phi^\dagger(y_n)]|0\rangle$$

N.B. $G(x_1, \ldots, x_n, y_1, \ldots, y_m) = 0$ for $m \neq n$

B. Propagator

1. *Definition:*

$$\begin{aligned}
iG_0(x,x') &= \langle 0|T[\phi(x)\phi^\dagger(x')]|0\rangle \\
&= \int \tilde{d}k\tilde{d}q\langle 0|T[(a(\boldsymbol{k})e^{-ikx}+b^\dagger(\boldsymbol{k})e^{ikx})(a^\dagger(\boldsymbol{q})e^{iqx'}+b(\boldsymbol{q})e^{-iqx'})]|0\rangle \\
&= \int \tilde{d}k\tilde{d}q[\Theta(t-t')\langle 0|a(\boldsymbol{k})a^\dagger(\boldsymbol{q})|0\rangle e^{iqx'-ikx}+\Theta(t'-t)\langle 0|b(\boldsymbol{q})b^\dagger(\boldsymbol{k})|0\rangle e^{ikx-iqx'}] \\
&= \int \tilde{d}k\tilde{d}q[\Theta(t-t')\langle 0|a^\dagger(\boldsymbol{q})a(\boldsymbol{k})+(2\pi^3 2\omega_{\boldsymbol{q}})\delta(\boldsymbol{q}-\boldsymbol{k})|0\rangle e^{iqx'-ikx} \\
&\quad +\Theta(t'-t)\langle 0|b^\dagger(\boldsymbol{k})b(\boldsymbol{q})+(2\pi^3 2\omega_{\boldsymbol{q}})\delta(\boldsymbol{q}-\boldsymbol{k})|0\rangle e^{ikx-iqx'}] \\
&= \int \tilde{d}k[\Theta(t-t')e^{-ik(x-x')}+\Theta(t'-t)e^{ik(x-x')}].
\end{aligned}$$

2. *Interpretation:* $\phi \approx a+b^\dagger$, $\phi^\dagger \approx b+a^\dagger$:

$G(x,y)$ is the transition amplitude of a particle (anti-particle) for $x^0 > y^0$ ($x^0 < y^0$)

3. *E.O.M.:*

$$(\Box_x+m^2)T[\phi(x)\phi^\dagger(y)] = -i\delta(x-y)\mathbb{1} + \mathrm{mass-shell\ contributions} \quad (\Box+m^2=-p^2+m^2=0)$$

Proof:

$$\begin{aligned}
&(\Box_x+m^2)T[\phi(x)\phi^\dagger(y)] \\
&= (\Box_x+m^2)[\Theta(x^0-y^0)\phi(x)\phi^\dagger(y)+\Theta(y^0-x^0)\phi^\dagger(y)\phi(x)] \\
&= T[(-\Delta_{\boldsymbol{x}}+m^2)\phi(x)\phi^\dagger(y)]+\partial^2_{x^0}[\Theta(x^0-y^0)\phi(x)\phi^\dagger(y)+\Theta(y^0-x^0)\phi^\dagger(y)\phi(x)] \\
&= T[(-\Delta_{\boldsymbol{x}}+m^2)\phi(x)\phi^\dagger(y)]+\partial_{x^0}\{T[\partial_0\phi(x)\phi^\dagger(y)]+\delta(x^0-y^0)[\phi(x)\phi^\dagger(y)-\phi^\dagger(y)\phi(x)]\} \\
&= T[(-\Delta_{\boldsymbol{x}}+m^2)\phi(x)\phi^\dagger(y)]+\partial_{x^0}\{T[\partial_0\phi(x)\phi^\dagger(y)]+\delta(x^0-y^0)[\phi(x),\phi^\dagger(y)]\} \\
&= T[(-\Delta_{\boldsymbol{x}}+m^2)\phi(x)\phi^\dagger(y)]+\partial_{x^0}T[\partial_0\phi(x)\phi^\dagger(y)] \\
&= T[(-\Delta_{\boldsymbol{x}}+m^2)\phi(x)\phi^\dagger(y)]+T[\partial_0^2\phi(x)\phi^\dagger(y)]+\delta(x^0-y^0)[\underbrace{\partial_0\phi(x)}_{\frac{\partial L}{\partial\partial_0\phi^\dagger}=\pi^\dagger},\phi^\dagger(y)] \\
&= T[\underbrace{(\Box_x+m^2)\phi(x)}_{=0\ \text{(E.O.M.)}}\phi^\dagger(y)]-i\delta(x-y)\mathbb{1} \\
&= -i\delta(x-y)\mathbb{1} \quad \leftarrow \quad [\pi^\dagger(x),\phi^\dagger(y)] = -i\delta(\boldsymbol{x}-\boldsymbol{y})
\end{aligned}$$

$\uparrow$ $\nwarrow$

operator for functions Fock space operator

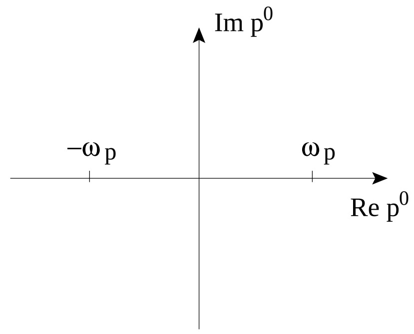
4. *Vacuum expectation value of the E.O.M.:*

$$(\Box_x+m^2)\langle 0|T[\phi(x)\phi^\dagger(y)]|0\rangle = (\Box_x+m^2)iG(x,y) = -i\delta(x-y)$$

5. *Solution:*

$$
\begin{aligned}
G(x,y) &= \int \frac{dp}{(2\pi)^4} e^{-ip(x-y)} G(p) \\
(\Box_x + m^2) i G(x,y) &= \int \frac{dp}{(2\pi)^4} e^{-ip(x-y)} (-p^2 + m^2) G(p) = -i\delta(x-y) = -i \int \frac{dp}{(2\pi)^4} e^{-ip(x-y)} \\
G(x,y) &= -(\Box + m^2)^{-1} = \int \frac{dp}{(2\pi)^4} \frac{e^{-ip(x-y)}}{p^2 - m^2} \quad \leftarrow \quad \Box \sim -p^2
\end{aligned}
$$

6. *Singularities:* mass-shell, $p^2 = m^2$



Way to avoid:

- Mathematics: Infinitesimal deformation of the contour of integration, $\omega_{\boldsymbol{p}} \to \omega_{\boldsymbol{p}} \pm i\epsilon$
- Physics:
  - (a) QFT: interaction with radiation $\implies$ radiation loss $\implies$ irreversibility
  - (b) $\not{T}$: non-perturbative, can not be left for radiative corrections $\implies$ break by H_0
  - (c) phase transitions $\implies$ infinitesimal $\not{T}$
  - (d) infinitesimal $\not{T}$:

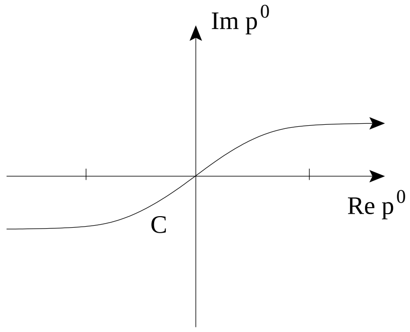
$$
e^{-it\omega_{\boldsymbol{p}}} \to \begin{cases} e^{-it(\omega_{\boldsymbol{p}} - i\epsilon)} & \text{particle}, \quad t > 0 \\ e^{-it(-\omega_{\boldsymbol{p}} + i\epsilon)} & \text{anti}-\text{particle}, \quad t < 0 \end{cases}
$$

$$
\omega_{\boldsymbol{p}} = \sqrt{m^2 + \boldsymbol{p}^2} \to \sqrt{m^2 + \boldsymbol{p}^2 - i\epsilon'} = \sqrt{m^2 + \boldsymbol{p}^2}\sqrt{1 - i\frac{\epsilon'}{m^2 + \boldsymbol{p}^2}} = \omega_{\boldsymbol{p}} - i\epsilon
$$

7. Feynman propagator:

   (a) Scalar particle:

$$
G(x,y) = \int \frac{dp}{(2\pi)^4} \frac{e^{-ip(x-y)}}{p^2 - m^2 + i\epsilon}
$$



(b) Gauge boson:

$$\begin{aligned} L &= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{m^2}{2}A^\mu A_\mu - \frac{\lambda}{2}(\partial^\mu A_\mu)^2 \\ iG_{\mu\nu}(x,y) &= \langle 0|T[A_\mu(x)A_\nu(y)]|0\rangle \\ &= -i\int \frac{dp}{(2\pi)^4}e^{-ip(x-y)}\left(\frac{g_{\mu\nu}-\frac{p_\mu p_\nu}{m^2}}{p^2-m^2+i\epsilon} + \frac{\frac{p_\mu p_\nu}{m^2}}{p^2-\frac{m^2}{\lambda^2}+i\epsilon}\right) \quad \leftarrow \quad \mathcal{O}\left(p^0\right)!!! \end{aligned}$$

(c) Fermion:

$$\begin{aligned} L &= \bar\psi[i\not\partial - m]\psi \\ iG_{\alpha\beta}(x,y) &= \langle 0|T[\psi_\alpha(x)\bar\psi_\beta(y)]|0\rangle \\ &= i\int \frac{dp}{(2\pi)^4}\frac{e^{-ip(x-y)}}{\not p - m + i\epsilon} \\ &= i\int \frac{dp}{(2\pi)^4}e^{-ip(x-y)}\frac{\not p + m}{p^2-m^2+i\epsilon} \end{aligned}$$

$$\nearrow$$

$$(\not p + m)(\not p - m) = p^2 - m^2$$

C. Weak form of Wick theorem

1. **Perturbation expansion:**

$$\begin{aligned} |i\rangle &= \phi_i(y_1)\cdots\phi_i(y_{m_i})|0\rangle, \quad |f\rangle = \phi_f(z_1)\cdots\phi_f(z_{m_f})|0\rangle \\ \mathcal{A} &= \langle 0|\phi_f(z_1)\cdots\phi_f(z_{m_f})T[e^{-i\int dtH_i(t)}]\phi_i(y_1)\cdots\phi_i(y_{m_i})|0\rangle \\ &= Z^{-\frac{1}{2}(n_i+n_f)}\langle 0|T[\phi(z_1)\cdots\phi(z_{m_f})e^{-i\int dtH_i(t)}\phi(y_1)\cdots\phi(y_{m_i})]|0\rangle \\ &= \langle 0|T[\phi_f(z_1)\cdots\phi_f(z_{m_f})e^{-\frac{ig}{4!}\int d^4x\phi^4(x)}\phi_f(y_1)\cdots\phi_f(y_{m_i})]|0\rangle \quad \leftarrow Z=1 \\ &= \sum_{n=0}^\infty \frac{1}{n!}\left(\frac{-ig}{4!}\right)^n\int dx_1\cdots dx_n \underbrace{\langle 0|T[\phi_f(z_1)\cdots\phi_f(z_{m_f})\phi^4(x_1)\cdots\phi^4(x_n)\phi_f(y_1)\cdots\phi_f(y_{m_i})]|0\rangle}_{\textit{in terms of free propagator}} \end{aligned}$$

2. **Simple example:** $A = \langle 0|a(\boldsymbol{q}_1)a(\boldsymbol{q}_2)a^\dagger(\boldsymbol{p}_1)a^\dagger(\boldsymbol{p}_2)|0\rangle$

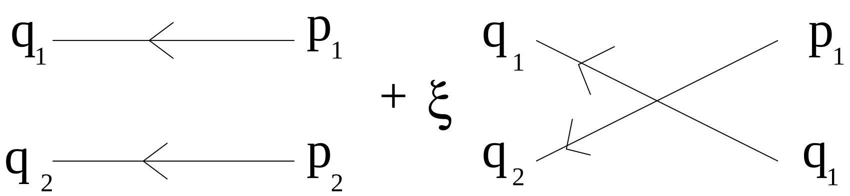
   Partnership of operators: pairing

(a) A : vacuum $\to$ vacuum

(b) $\overbrace{\quad}$: partners, one creates the other removes an excitation

(c) non-interacting excitations $\implies$ product of contributions

$$
\begin{aligned}
A &= \langle 0|a(\boldsymbol{q}_1)\overbrace{a(\boldsymbol{q}_2)a^\dagger(\boldsymbol{p}_2)}^{\boldsymbol{q}_2=\boldsymbol{p}_2}a^\dagger(\boldsymbol{p}_1)|0\rangle \\
&= \langle 0|a(\boldsymbol{q}_1)[\xi a^\dagger(\boldsymbol{p}_2)a(\boldsymbol{q}_2)+\overbrace{(2\pi)^3 2\omega_{\boldsymbol{p}_2}\delta(\boldsymbol{p}_2-\boldsymbol{q}_2)}]a^\dagger(\boldsymbol{p}_1)|0\rangle \\
&= \xi\langle 0|a(\boldsymbol{q}_1)a^\dagger(\boldsymbol{p}_2)\overbrace{a(\boldsymbol{q}_2)a^\dagger(\boldsymbol{p}_1)}^{\boldsymbol{q}_2=\boldsymbol{p}_1}|0\rangle+\overbrace{(2\pi)^3 2\omega_{\boldsymbol{p}_2}\delta(\boldsymbol{p}_2-\boldsymbol{q}_2)}\langle 0|\overbrace{a(\boldsymbol{q}_1)a^\dagger(\boldsymbol{p}_1)}^{\boldsymbol{q}_1=\boldsymbol{p}_1}|0\rangle \\
&= \xi\langle 0|\overbrace{a(\boldsymbol{q}_1)a^\dagger(\boldsymbol{p}_2)}^{\boldsymbol{q}_1=\boldsymbol{p}_2}[\xi a^\dagger(\boldsymbol{p}_1)a(\boldsymbol{q}_2)+\overbrace{(2\pi)^3 2\omega_{\boldsymbol{p}_1}\delta(\boldsymbol{p}_1-\boldsymbol{q}_2)}]|0\rangle \\
&\quad +\overbrace{(2\pi)^3 2\omega_{\boldsymbol{p}_2}\delta(\boldsymbol{p}_2-\boldsymbol{q}_2)}\langle 0|[\xi a^\dagger(\boldsymbol{p}_1)a(\boldsymbol{q}_1)+\overbrace{(2\pi)^3 2\omega_{\boldsymbol{p}_1}\delta(\boldsymbol{p}_1-\boldsymbol{q}_1)}]|0\rangle \\
&= \xi\,\overbrace{(2\pi)^3 2\omega_{\boldsymbol{p}_1}\delta(\boldsymbol{p}_1-\boldsymbol{q}_2)}\langle 0|\xi a^\dagger(\boldsymbol{p}_2)a(\boldsymbol{q}_1)+\overbrace{(2\pi)^3 2\omega_{\boldsymbol{p}_2}\delta(\boldsymbol{p}_2-\boldsymbol{q}_1)}]|0\rangle \\
&\quad +\overbrace{(2\pi)^3 2\omega_{\boldsymbol{p}_2}\delta(\boldsymbol{p}_2-\boldsymbol{q}_2)}\langle 0|[\xi a^\dagger(\boldsymbol{p}_1)a(\boldsymbol{q}_1)+\overbrace{(2\pi)^3 2\omega_{\boldsymbol{p}_1}\delta(\boldsymbol{p}_1-\boldsymbol{q}_1)}]|0\rangle \\
&= \xi\,\overbrace{(2\pi)^3 2\omega_{\boldsymbol{p}_2}\delta(\boldsymbol{p}_2-\boldsymbol{q}_1)}\,\overbrace{(2\pi)^3 2\omega_{\boldsymbol{p}_1}\delta(\boldsymbol{q}_1-\boldsymbol{p}_1)}+\overbrace{(2\pi)^3 2\omega_{\boldsymbol{p}_2}\delta(\boldsymbol{p}_2-\boldsymbol{q}_2)}\,\overbrace{(2\pi)^3 2\omega_{\boldsymbol{p}_2}\delta(\boldsymbol{q}_1-\boldsymbol{p}_1)}
\end{aligned}
$$



3. **Time ordered product:**

Partnership of operators: pairing

(a) A : vacuum $\to$ vacuum

(b) $\overbrace{\quad}$: partners, one creates the other removes an excitation

i. Pair of operators A, B: $[A,B]_\xi = c(A,B)\mathbb{1}$

ii. We need:

A. $\langle 0|\overbrace{AB}|0\rangle = \langle 0|T[AB]|0\rangle$ to find the propagator

B. $\overbrace{AB} = f(A,B)\mathbb{1}$, $f(A,B)$ being a c-number: to factorize the propagator

iii. Solution: $\overbrace{AB} = T[AB] - :AB:$

A. $\langle 0|T[AB] - :AB:|0\rangle = \langle 0|T[AB]|0\rangle$ since $\langle 0|:AB:|0\rangle = 0$

B. $\overbrace{AB} = T[AB] - :AB: := \begin{cases} \pm(AB - AB) = 0\mathbb{1} \\ \pm(AB - BA) = \pm c(A,B)\mathbb{1} \end{cases}$

(c) non-interacting excitations $\implies$ product of contributions

Example: $\phi(x) = A(x) + B^\dagger(x)$, $\overbrace{A(x)B(y)} = \overbrace{A^\dagger(x)B^\dagger(y)} = 0$

$$\overbrace{\phi(x)\phi^\dagger(y)} = \overbrace{[A(x)+B^\dagger(x)][A^\dagger(y)+B(y)]} = \overbrace{A(x)A^\dagger(y)} + \overbrace{B^\dagger(x)B(y)},$$

$$\begin{aligned}
i^2G(x_1,x_2,y_1,y_2) &= \langle 0|T[\phi(x_1)\phi(x_2)\phi^\dagger(y_1)\phi^\dagger(y_2)]|0\rangle \\
&= \langle 0|T[\phi(x_1)[\overbrace{\phi(x_2)\phi^\dagger(y_1)} + :\phi(x_2)\phi^\dagger(y_1):]\phi^\dagger(y_2)]|0\rangle \\
&= \langle 0|\overbrace{\phi(x_1)\phi^\dagger(y_2)} + :\phi(x_1)\phi^\dagger(y_2):|0\rangle\langle 0|\overbrace{\phi(x_2)\phi^\dagger(y_1)}|0\rangle + \langle 0|T[\phi(x_1):\phi(x_2)\phi^\dagger(y_1):\phi^\dagger(y_2)] \\
&= \langle 0|\overbrace{\phi(x_1)\phi^\dagger(y_2)}|0\rangle\langle 0|\overbrace{\phi(x_2)\phi^\dagger(y_1)}|0\rangle + \langle 0|T[\phi(x_1):\phi(x_2)\phi^\dagger(y_1):\phi^\dagger(y_2)]|0\rangle \\
&= \langle 0|\overbrace{\phi(x_1)\phi^\dagger(y_2)}|0\rangle\langle 0|\overbrace{\phi(x_2)\phi^\dagger(y_1)}|0\rangle \\
&\quad + \langle 0|T[\underbrace{\phi(x_1)}_{A+B^\dagger}[\xi A^\dagger(y_1)A(x_2) + A(x_2)B(y_1) + B^\dagger(x_2)A^\dagger(y_1) + B^\dagger(x_2)B(y_1)]\underbrace{\phi^\dagger(y_2)}_{A^\dagger+B}]|0\rangle \\
&= \langle 0|\overbrace{\phi(x_1)\phi^\dagger(y_2)}|0\rangle\langle 0|\overbrace{\phi(x_2)\phi^\dagger(y_1)}|0\rangle \\
&\quad + \xi\langle 0|\overbrace{A(x_1)A^\dagger(y_1)}|0\rangle\langle 0|\overbrace{A(x_2)A^\dagger(y_2)}|0\rangle + \xi\langle 0|\overbrace{B^\dagger(x_1)B(y_1)}|0\rangle\langle 0|\overbrace{A(x_2)A^\dagger(y_2)}|0\rangle \\
&\quad + \xi\langle 0|\overbrace{A(x_1)A^\dagger(y_1)}|0\rangle\langle 0|\overbrace{B^\dagger(x_2)B(y_2)}|0\rangle + \xi\langle 0|\overbrace{B^\dagger(x_1)B(y_1)}|0\rangle\langle 0|\overbrace{B^\dagger(x_2)B(y_2)}|0\rangle \\
&= \langle 0|\overbrace{\phi(x_1)\phi^\dagger(y_2)}|0\rangle\langle 0|\overbrace{\phi(x_2)\phi^\dagger(y_1)}|0\rangle + \xi\langle 0|\overbrace{\phi(x_1)\phi^\dagger(y_1)}|0\rangle\langle 0|\overbrace{\phi(x_2)\phi^\dagger(y_2)}|0\rangle
\end{aligned}$$

4. **Wick's theorem:**

$$\begin{aligned}
\langle 0|T[\phi(x_1)\cdots\phi(x_n)\phi^\dagger(y_1)\cdots\phi^\dagger(y_n)]|0\rangle &= i^nG_0(x_1,\ldots,x_n,y_1,\ldots,y_n) \\
&= \sum_{\pi\in S_n}\xi^{\sigma_\pi}\langle 0|T[\phi(x_1)\phi^\dagger(y_{\pi(1)})]|0\rangle\cdots\langle 0|T[\phi(x_n)\phi^\dagger(y_{\pi(n)})]|0\rangle \\
&= \sum_{\pi\in S_n}\xi^{\sigma_\pi}iG_0(x_1-y_{\pi(1)})\cdots iG_0(x_n-y_{\pi(n)}) \\
\langle 0|T[\phi(x_1)\cdots\phi(x_2n)]|0\rangle &= i^nG(x_1,\ldots,x_{2n}) \\
&= \frac{1}{n!2^n}\sum_{\pi\in S_{2n}}\langle 0|T[\phi(x_{\pi(1)})\phi(x_{\pi(2)})]|0\rangle\cdots\langle 0|T[\phi(x_{\pi(2n-1)})\phi(x_{\pi(2n)})]|0\rangle \\
&= \frac{i^n}{n!2^n}\sum_{\pi\in S_{2n}}G(x_{\pi(1)}-x_{\pi(2)})\cdots G(x_{\pi(2n-1)}-x_{\pi(2n)})
\end{aligned}$$

($n!$: possible order of the n pairs of space-time points, 2^n: double counting of each pair)

D. Scalar field theory

$$L=\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-\frac{m^2}{2}\phi^2-\frac{g}{4!}\phi^4$$

1. **Perturbation expansion:**

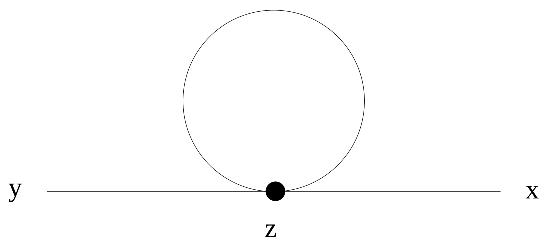
$$\begin{aligned}\langle 0|T[\phi(y)S\phi(x)]|0\rangle &= \langle 0|T[\phi(y)e^{-i\frac{g}{4!}\int dz\phi^4(z)}\phi(x)]|0\rangle\\ &= \sum_{n=0}^{\infty}\left(\frac{-ig}{4!}\right)^n\frac{1}{n!}\int dz_1\cdots dz_n\langle 0|T[\phi(y)\phi^4(z_1)\cdots\phi^4(z_n)\phi(x)]|0\rangle\\ &= \langle 0|T[\phi(y)\phi(x)]|0\rangle-\frac{ig}{4!}\int dz\langle 0|T[\phi(y)\phi^4(z)\phi(x)]|0\rangle\\ &\quad+\frac{(-ig)^2}{2(4!)^2}\int dz_1dz_2\langle 0|T[\phi(y)\phi^4(z_1)\phi^4(z_2)\phi(x)]|0\rangle+\mathcal{O}\left(g^3\right).\end{aligned}$$

2. $\mathcal{O}\left(g^0\right)$**:**

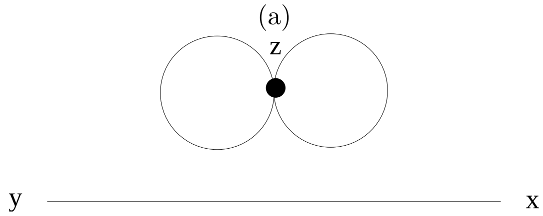
$$\langle 0|T[\phi(y)S\phi(x)]|0\rangle^{(0)}=\langle 0|T[\phi(y)\phi(x)]|0\rangle=iG(x,y).$$

3. $\mathcal{O}\left(g\right)$**:**

$$\begin{aligned}\langle 0|T[\phi(y)S\phi(x)]|0\rangle^{(1)} &= -\frac{ig}{4!}\int dz\langle 0|T[\phi(y)\phi^4(z)\phi(x)]|0\rangle\\ &= \underbrace{-\frac{ig}{2}\int dziG(y,z)iG(z,z)iG(z,x)}_{(a)}\underbrace{-\frac{ig}{8}\int dziG(y,x)iG(z,z)iG(z,z)}_{(b)}\end{aligned}$$



(a)



(b)

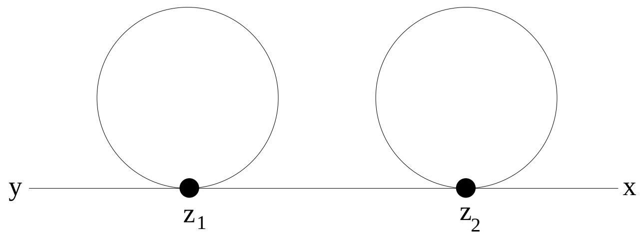
4. **Momentum space:**

$$
\begin{aligned}
G_a^{(1)}(p) &= \int dy e^{-ip(y-x)} \langle 0|T[\phi(y)S\phi(x)]|0\rangle_a^{(1)} \\
&= -\frac{g}{2}\int dy e^{-ip(y-x)} \int dz \frac{dp_1}{(2\pi)^4}\frac{dp_2}{(2\pi)^4}\frac{dp_3}{(2\pi)^4} G(p_1)G(p_2)G(p_3)e^{ip_1(y-z)+ip_2(z-z)+ip_3(z-x)} \\
&= -\frac{g}{2}\int dy e^{-ip(y-x)} \int \frac{dp_1}{(2\pi)^4}\frac{dp_2}{(2\pi)^4}\frac{dp_3}{(2\pi)^4}(2\pi)^4\delta(p_1-p_3) G(p_1)G(p_2)G(p_3)e^{ip_1 y - ip_3 x} \\
&= -\frac{g}{2}G^2(p)\int \frac{dp_2}{(2\pi)^4}G(p_2) \\
&= \frac{g}{2}\frac{1}{(p^2-m^2+i\epsilon)^2}\int \frac{dp_2}{(2\pi)^4}\frac{1}{p_2^2-m^2+i\epsilon}
\end{aligned}
$$

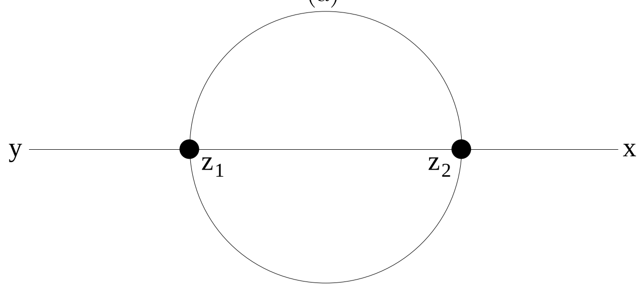
5. **Disconnected bubble graphs drop:**

$$
\begin{aligned}
iG_{\text{int}}(y,x) &= \frac{\langle 0|T[\phi(y)S\phi(x)]|0\rangle}{\langle 0|S|0\rangle} \\
&= \frac{\langle 0|T[\phi(y)\phi(x)]|0\rangle - \frac{ig}{4!}\int dz \langle 0|T[\phi(y)\phi^4(z)\phi(x)]|0\rangle}{\langle 0|0\rangle - \frac{ig}{4!}\int dz \langle 0|\phi^4(z)|0\rangle} + \mathcal{O}\left(g^2\right) \\
&= \langle 0|T[\phi(y)\phi(x)]|0\rangle - \frac{ig}{4!}\int dz \langle 0|T[\phi(y)\phi^4(z)\phi(x)]|0\rangle \\
&\quad + \frac{ig}{4!}\langle 0|T[\phi(y)\phi(x)]|0\rangle \int dz \langle 0|\phi^4(z)|0\rangle + \mathcal{O}\left(g^2\right) \\
&= \langle 0|T[\phi(y)\phi(x)]|0\rangle - \frac{ig}{4!}\int dz \langle 0|T[\phi(y)\phi^4(z)\phi(x)]|0\rangle_a + \mathcal{O}\left(g^2\right).
\end{aligned}
$$

6. **Connected $\mathcal{O}\left(g^2\right)$ graphs:**



(a)



(b)

$$\langle 0|T[\phi(y)S\phi(x)]|0\rangle^{(2)} = \frac{(-ig)^2}{2(4!)^2}\int dz_1dz_2\langle 0|T[\phi(y)\phi^4(z_1)\phi^4(z_2)\phi(x)]|0\rangle$$
$$= \underbrace{\frac{(-ig)^2}{4}\int dz_1dz_2 iG(y-z_1)iG(z_1-z_1)iG(z_1-z_2)iG(z_2-z_2)iG(z_2-x)}_{(a)}$$
$$\underbrace{+\frac{(-ig)^2}{3!}\int dz_1dz_2 iG(y-z_1)[iG(z_1-z_1)]^3 iG(z_2-x)}_{(b)},$$

Momentum space:

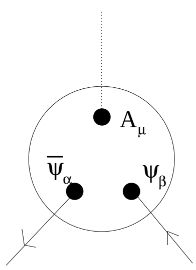
$$iG_a^{(2)}(p) = \int dy e^{-ip(y-x)}\langle 0|T[\phi(y)S\phi(x)]|0\rangle_a^{(2)}$$
$$= -\frac{ig^2}{4}\int dy e^{-ip(y-x)}\int dz_1dz_2\frac{dp_1}{(2\pi)^4}\cdots\frac{dp_5}{(2\pi)^4}G(p_1)G(p_2)G(p_3)G(p_4)G(p_5)$$
$$\times e^{ip_1(y-z_1)+ip_2(z_1-z_1)+ip_3(z_1-z_2)+ip_2(z_2-z_2)+ip_3(z_2-x)}$$
$$= -\frac{ig}{4}G^3(p)\left[\int\frac{dp_2}{(2\pi)^4}G(p_2)\right]^2$$
$$iG_b^{(2)}(p) = \int dy e^{-ip(y-x)}\langle 0|T[\phi(y)S\phi(x)]|0\rangle_b^{(2)}$$
$$= -\frac{ig^2}{4}\int dy e^{-ip(y-x)}\int dz_1dz_2\frac{dp_1}{(2\pi)^4}\cdots\frac{dp_5}{(2\pi)^4}G(p_1)G(p_2)G(p_3)G(p_4)G(p_5)$$
$$\times e^{ip_1(y-z_1)+ip_2(z_1-z_2)+ip_3(z_1-z_2)+ip_2(z_1-z_2)+ip_3(z_2-x)}$$
$$= -\frac{ig}{4}G^2(p)\int\frac{dp_2}{(2\pi)^4}\frac{dp_3}{(2\pi)^4}G(p_2)G(p_3)G(p-p_2-p_3)$$

E. QED

$$L = \bar\psi[i\gamma^\mu(\partial_\mu - ieA_\mu) - m]\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{\lambda}{2}(\partial^\mu A_\mu)^2, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

1. **Perturbation expansion:**

$$H_i = e\int d^3x\bar\psi\gamma^\mu\psi A_\mu.$$



2. **Elastic $e^-e^- \to e^-e^-$ scattering:**

- *Scattering amplitude:*

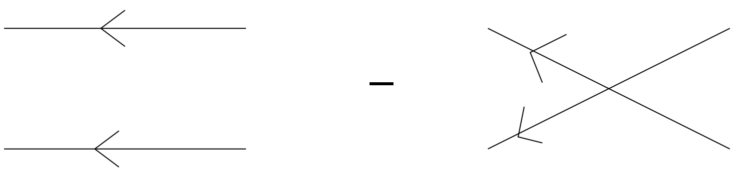
$$\begin{aligned}
\mathcal{A} &= \langle 0|\psi_{f\beta_1}(z_1)\psi_{f\beta_2}(z_2)S\psi_{i\alpha_1}(y_1)\psi_{f\alpha_2}(y_2)|0\rangle_0 \\
&= \langle 0|\psi_{f\beta_1}(z_1)\psi_{f\beta_2}(z_2)T[e^{-ie\int dx\bar\psi(x)\gamma^\mu\psi(x)A_\mu(x)}]\psi_{i\alpha_1}(y_1)\psi_{f\alpha_2}(y_2)|0\rangle_0 \\
&= \sum_{n=0}^\infty \frac{(-ie)^n}{n!}\int dx_1\cdots dx_n \\
&\quad \times\langle 0|T[\psi_{\beta_1}(z_1)\psi_{\beta_2}(z_2)\bar\psi(x_1)\gamma^{\mu_1}\psi(x_1)A_{\mu_1}(x_1)\cdots\bar\psi(x_n)\gamma^{\mu_n}\psi(x_n)A_{\mu_n}(x_n)\psi_{\alpha_1}(y_1)\psi_{\alpha_2}(y_2)]|0\rangle_0.
\end{aligned}$$

- *Factorization of the electron and the photon Fock spaces:*

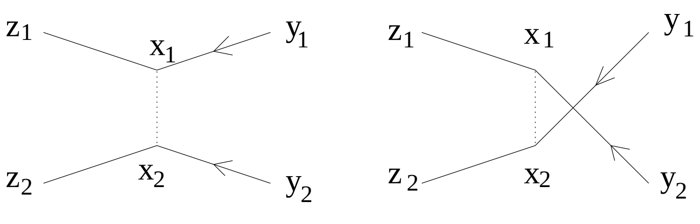
$$\begin{aligned}
\mathcal{A} &= \sum_{n=0}^\infty \frac{(-ie)^n}{n!}\int dx_1\cdots dx_n\langle 0|T[A_{\mu_1}(x_1)\cdots A_{\mu_n}(x_n)]|0\rangle_0 \\
&\quad \times\langle 0|T[\psi_{\beta_1}(z_1)\psi_{\beta_2}(z_2)\bar\psi(x_1)\gamma^{\mu_1}\psi(x_1)\cdots\bar\psi(x_n)\gamma^{\mu_n}\psi(x_n)\bar\psi_{\alpha_1}(y_1)\bar\psi_{\alpha_2}(y_2)]|0\rangle_0 \\
&= \langle 0|T[\psi_{\beta_1}(z_1)\psi_{\beta_2}(z_2)\bar\psi_{\alpha_1}(y_1)\bar\psi_{\alpha_2}(y_2)]|0\rangle_0 \\
&\quad +\frac{(-ie)^2}{2!}\int dx_1dx_2\langle 0|T[A_{\mu_1}(x_1)A_{\mu_2}(x_2)]|0\rangle_0 \\
&\quad \times\langle 0|T[\psi_{\beta_1}(z_1)\psi_{\beta_2}(z_2)\bar\psi(x_1)\gamma^{\mu_1}\psi(x_1)\bar\psi(x_2)\gamma^{\mu_2}\psi(x_2)\bar\psi_{\alpha_1}(y_1)\bar\psi_{\alpha_2}(y_2)]|0\rangle_0+\mathcal{O}\left(e^4\right).
\end{aligned}$$

- $\mathcal{O}\left(e^0\right)$*:*

$$\langle 0|T[\psi_{\beta_1}(z_1)\psi_{\beta_2}(z_2)\bar\psi_{\alpha_1}(y_1)\bar\psi_{\alpha_2}(y_2)]|0\rangle_0$$



- $\mathcal{O}\left(e^2\right)$*:*



$$\begin{aligned}
&-e^2\int dx_1dx_2iD_{\mu_1\mu_2}(x_1-x_2)iG_{\beta_1\kappa_1}(z_1-x_1)\gamma^{\mu_1}_{\kappa_1\rho_1}iG_{\rho_1\alpha_1}(x_1-y_1)iG_{\beta_1\kappa_1}(z_2-x_2)\gamma^{\mu_2}_{\kappa_2\rho_2}iG_{\rho_2\alpha_2}(x_2-y_2) \\
&+e^2\int dx_1dx_2iD_{\mu_1\mu_2}(x_1-x_2)iG_{\beta_1\kappa_1}(z_1-x_1)\gamma^{\mu_1}_{\kappa_1\rho_1}iG_{\rho_1\alpha_1}(x_1-y_2)iG_{\beta_1\kappa_1}(z_2-x_2)\gamma^{\mu_2}_{\kappa_2\rho_2}iG_{\rho_2\alpha_2}(x_2-y_1).
\end{aligned}$$

3. **Feynman rules:**

- *Definition of the process:*

(a) Initial state: $(n_{i-} \times e^-) + (n_{i+} \times e^+) + (n_{i0} \times \gamma)$

(b) Final state: $(n_{f-} \times e^-) + (n_{f+} \times e^+) + (n_{f0} \times \gamma)$

(c) $\mathcal{O}\left(e^\ell\right)$

- *Place symbols on the paper:*
  - External legs

$$\left(\prod_{j=1}^{n_{i0}} A_{\mu_j^{i0}}(y_j^{i0})\right)\left(\prod_{j=1}^{n_{i+}} \psi_{\alpha_j^{i+}}(y_j^{i+})\right)\left(\prod_{j=1}^{n_{i-}} \bar{\psi}_{\alpha_j^{i-}}(y_j^{i-})\right)$$
$$\left(\prod_{j=1}^{n_{f0}} A_{\mu_j^{f0}}(y_j^{f0})\right)\left(\prod_{j=1}^{n_{f+}} \bar{\psi}_{\alpha_j^{f+}}(y_j^{f+})\right)\left(\prod_{j=1}^{n_{f-}} \psi_{\alpha_j^{f-}}(y_j^{f-})\right)$$

  - Vertices:

$$\prod_{k=1}^{\ell} A_{\mu^k}(z_k)\bar{\psi}(z_k)\gamma^{\mu_k}\psi(z_k)$$

- *List all (topologically) different pairing:*

  Paired symbols are connected by lines (propagators):

  - Electron (positron):

$$\langle 0|T[\psi_a(x)\bar{\psi}_b(y)]|0\rangle = iG_{a,b}(x,y))$$

  - Photon:

$$\langle 0|T[A_\mu(x)A_\nu(y)]|0\rangle = iD_{\mu,\nu}(x,y))$$

- *Integration over verticex position:*

$$\prod_{k=1}^{\ell}\int dz_k$$

- *Prefactor:*

number of closed fermion loops

$\downarrow$

$$\frac{(-ie)^\ell}{\ell!}(-1)^N S_G$$

$\uparrow$

number of (topological) identical graphs

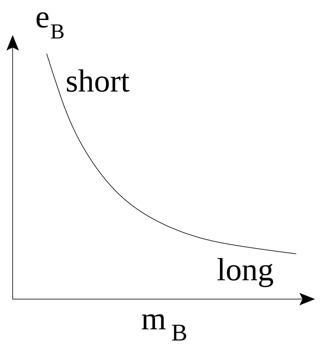
F. Renormaliaztion

1. **Fixing QED:**

$$
\begin{aligned}
L &= \bar{\psi}[i\gamma^\mu(\partial_\mu - ie_B A_\mu) - m_B]\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \\
P_1 &= F_1(m_B, e_B, \Lambda) \\
P_2 &= F_2(m_B, e_B, \Lambda)
\end{aligned}
$$

2. **Renormalized trajectory:** physical parameters at the scale Λ

$$m_B = m_B(\Lambda), \quad e_B = e_B(\Lambda)$$



3. **Renormalizable theory:** $\lim_{\Lambda\to\infty}$ exists

4. **Non-renormalizable theory:**

   - *Fixing conditions are not invertible beyond a certain cutoff:* There is a maximal Λ_L (Landau pole)
   - *Strong interactions at short distances:* $g_B(\Lambda_L) = \infty$

5. **Asymptotical freedom:** Renormalizable theory with $\lim_{\Lambda\to\infty} g_B(\Lambda) = 0$

   (Non-Abelien gauge theories)

6. **Non-asymptotical free:** There is always a Landau pole (numerical results only)

   ($U(1)$ and the Higgs sectors of SM)