# TOPOLOGICAL CHARGE OF THE LATTICE $CP^{N-1}$ MODEL

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ILL-(TH)-83-3
January 1983

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ABSTRACT

It is showed numerically that the path integral of the vacuum-to-vacuum amplitude of the lattice $CP^{n-1}$ model is dominated by quasiparticle like configurations in the continuum limit. The dispersion of the total quasiparticle number which is zero in any finite order of the weak coupling expansion on periodic lattice follows the expected continuum limit.

+ On leave from the Central Research Institute for Physics, Budapest, Hungary.
* Supported in part by the NSF under grant NSF PHY82-01948.

PACS Index:11.15.Tk
64.60.Cn

Typed by Cindy Elder

## 1. INTRODUCTION

Some progress has been achieved in the numerical study of quantum field theories by applying Monte Carlo simulation to field theories with lattice regularisation. The apparent disadvantage of this method is that it is purely numerical, one can not easily learn physics directly from the configurations generated by the computer. It is natural to ask whether we can apply some of the results of the semiclassical approximation discovered in the continuum version of some theories [1,2]. The attractive feature of the semiclassical approximation is that by the assumption of dominant saddle point configurations in the path integral representation one can qualitatively understand some nonperturbative phenomena in terms of a "classical" instanton gas. In order to apply this strategy to lattice calculations one first has to describe the subset of dominant configurations. After that one can check whether this subclass dominates the configuration sequence generated by the Monte Carlo process.

A possible way of implementing such an approximation on the lattice is to try to generalise the procedure which has been developed in the continuum theory. The first step has been done in this direction [3]. Another possible way is to start from the beginning and study almost pure gauge configurations (in the case of gauge models) in order to extract information concerning the subset of dominant configurations. We propose such an approach in the case of the two dimensional lattice $CP^{n-1}$ model. In the continuum case the assumption of the finiteness of the action leads us to study pure gauge configurations at space-time infinity. As a result we find a series of subclasses characterised by an integer number, the topological charge. In our case the study of almost pure gauge configurations in the vicinity of a space-time point helps us to recognise the quasiparticle structure and to introduce the topological charge density.

Although our definition of the topological charge is formally equivalent with one given by Berg and Luscher [11] the motivation for the expression is different. In fact it gives new insight into the subset of the relevant configurations of the path integral.

We shall use the gauge covariant formulation of the $CP^{n-1}$ model with an auxiliary U(1) gauge field as proposed in ref. [4]. The dynamical variables are the n component complex vectors $z^i_{\vec{n}}$ of unit length attached to the lattice sites, and the U(1) $z^i_{\vec{n}}$ link variables $U_\mu(\vec{n})$. The action and the partition function are

$$Z = \int D[z]\, D[\bar{z}]\, D[U]\, \pi\, \delta\, (|z\bar{z}| - 1)\, \exp\{-S(z,\bar{z},U)\}, \tag{1a}$$

and

$$S = -\frac{n}{2g^2} \sum \{\bar{z}_{\vec{n}}\, U_\mu(\vec{n})\, z_{\vec{n}+\hat{\mu}} + c.c.\}, \tag{1b}$$

respectively.

The beta function of this lattice model was calculated in the large n limit. It allows us to introduce a scale parameter

$$\Lambda_L = a^{-1} e^{-\frac{\pi^2}{g^2}}, \tag{2}$$

which remains finite as the cutoff a is removed.

The central question of this study is whether the quasiparticle structure defined on the lattice remains relevant in the continuum limit. We carry out a Monte Carlo calculation for the topological susceptibility $\chi_t = \langle Q^2\rangle/V$ [3] (Q is the topological charge and V is the space-time volume). According to the dimension of $\chi_t$ (the volume in the denominator is to render $\chi_t$ to be finite in the large volume limit) we find that it is proportional to $\Lambda_L^2$.

The paper is organised as follows. Sect. 2 contains the definition of the topological charge density. The relation between the lattice and the continuum definitions is discussed in Sect.3. In Sect. 4 the result of the Monte Carlo calculation is presented. Sect. 5 contains the calculation of the the ratio $\Lambda_L/\Lambda_{PV}$ which is necessary to relate the topological susceptibility to other dimensional quantities. We comment on the effects of a genuine lattice mode of the theory in Sect. 6. Finally the summary is given in Sect. 7.

## 2. QUASIPARTICLES IN THE LATTICE $CP^{n-1}$ MODEL

In this section we shall study the local properties of field configurations preferred by the dynamics (1) in the continuum limit.

We introduce two phase variables which prove to be useful. One is the phase of the U(1) gauge field, the other is the phase of the scalar product of two neighbouring z fields:

$$e^{i A_\mu(\vec{n})} = U_\mu(\vec{n}), \tag{3a}$$

and

$$e^{i\phi_\mu(\vec{n})} = \frac{\bar{z}_{\vec{n}} \cdot z_{\vec{n}+\hat{\mu}}}{|\bar{z}_{\vec{n}} \cdot z_{\vec{n}+\hat{\mu}}|} . \tag{3b}$$

We require by definition that

$$-\pi \leqslant A_\mu \text{ and } \phi_\mu < \pi . \tag{4}$$

In terms of these variables the action (1) reads as

$$S = -\frac{n}{g^2} \sum |\bar{z}_{\vec{n}} z_{\vec{n}+\hat{\mu}}| \cos (A_\mu(\vec{n}) + \phi_\mu(\vec{n})).$$

The configuration $\{z_{\vec{n}}, A_\mu(\vec{n})\}$ is called irregular if

$$|\bar{z}_{\vec{n}} z_{\vec{n}+\hat{\mu}}| \cos \left(A_\mu(\vec{n}) + \phi_\mu(\vec{n})\right) < 1-\varepsilon,$$

for at least one link ($\varepsilon$ is a small positive number). We expect that in the continuum limit ($g^2 \to 0$) these configurations become suppressed.

The regular, relevant configurations are characterised by the inequalities

$$|\bar{z}_{\vec{n}} z_{\vec{n}+\hat{\mu}}|, \text{ and } \cos (A_\mu(\vec{n}) + \phi_\mu(\vec{n})) > 1 - \varepsilon/2, \tag{5}$$

for each link.

It is easy to see from condition (4) that the signed sum of our phase variables along a closed loop (e. g. curlA for a plaquette) is not gauge invariant. Those gauge transformations which will change the signed sum by an integer multiple of $2\pi$ will be called vortex like ones. An example of a vortex like gauge transformation is depicted on Fig. 1. Since the product of the $U_\mu(\vec{n})$ variables along a closed loop is strictly gauge invariant, the modulo $2\pi$ value of the sum of the phase variables (i. e. log ($\pi$U)) is strictly gauge invariant as well.

Let us study the implication of the first inequality of (5) for the field z at a plaquette. In order to simplify the situation we choose a gauge where three of the four $\phi_\mu(\vec{n})$ variables along the selected plaquette vanish. In this gauge the scalar products of the z's bounded by a link with $\phi = 0$ are real and

$$\sum_{i=1}^{2n} z^i_{\vec{n}} \, z^i_{\vec{n}+\hat{\mu}} > 1 - \varepsilon/2.$$

These inequalities show that the vectors $z^i_{\vec{n}}$ of 2n real component on the unit sphere ($\bar{z} \cdot z = 1$) are almost parallel. Consequently we have the relation

$$q'_{\vec{n}} = \text{curl}\ \phi = 2\pi\ P_{\vec{n}} + O(\varepsilon), \tag{6}$$

for the original configuration (without fixing the gauge). We call $P_{\vec{n}}$ the quasiparticle content of the plaquette $\vec{n}$. This number will change under vortex like gauge transformations.

We define the topological charge density and the total topological charge of the field z as

$$q_{\vec{n}} = \text{curl}\ \phi\ (\text{mod}\ 2\pi),\ -\pi \leq q_{\vec{n}} < \pi,$$

and

$$Q = (2\pi)^{-1} \sum_{\vec{n}} q_{\vec{n}};$$

respectively. In the case of periodic boundary conditions the total topological charge $Q = \sum P_{\vec{n}}$ is integer for arbitrary configuration. Consequently it is free from perturbative contributions on a finite lattice.

Now we show that a change of Q is impossible without passing through an irregular configuration apart from a subset of zero measure in (1). First we remark that, when a gauge transformation is applied along a plaquette, curl $\phi$ jumps by $2\pi$ if the variable $\phi$ takes the value $\pm\,\pi$. In this case the neighboring plaquette containing the same link shows an opposite jump in curl $\phi$. In other words the vortex like gauge transformation creates a pair of quasiparticles. Let us imagine a path in configuration space connecting two configurations with $Q = N$ and $Q = N+1$. Along this path we may have several points where quasiparticle pairs are created and we have at least one point where Q jumps. We omit those case where a quasiparticle pair is created at the same point where Q jumps. These points represent a subset of zero measure in the configuration space. In the remaining cases we have an interval on the path where Q jumps without creating quasiparticle pairs. It means that going through such a one parameter family of configurations we cross the $\pm\pi$ values by a curl $\phi$ without any $\phi_\mu(\vec{n})$ reaching $\pm\,\pi$. It is impossible to satisfy relation (6) during such a deformation.

The lesson of the previous argument is that we expect an enhancement of the configurations where curl $\phi$ is close to integer multiple of $2\pi$. This effect is similar to one of a term like $\beta' \sum \cos(\mathrm{curl}\phi)$ in the action.

It is worthwhile to note the role of the boundary condition in this construction. In the case of free boundary condition Q is not integer so $\langle Q^2\rangle$ may be divergent in the continuum limit and when a vortex like gauge transformation is applied at the boundary sum $\sum P_{\vec{n}}$ will be changed.

It is possible to define a similar integer number using the phase of the U(1) gauge field A. Because of the second inequality of (5) we expect a similar enhancement of configurations where the values of curlA are close to integer multiple of $2\pi$. Hence we define the topological charge assosciated to the gauge field U as

$$q^u_{\vec{n}} = \text{curl } A_\mu (\text{mod } 2\pi),$$

$$Q^u = (2\pi)^{-1} \sum q^u_{\vec{n}} .$$

3. QUASIPARTICLES AND THE TOPOLOGICAL CHARGE

The semicalssical approximation of (1) consist of expanding the integrand around its maxima. But unfortunately the extremal configurations are unknown at the present. In the special case when the lattice field approaches an instanton solution of the continuum model we shall show that the instanton appears as a quasiparticle on the lattice field configuration and the total topological charge of the continuum configuration is the total quasiparticle number $Q = \sum_{\vec{n}} P_{\vec{n}}$ .

Naturally there might be several other extremal configurations of the lattice theory which do not fit to the instanton picture of the continuum theory, or it might happen that we have no dominant configurations at all and the path integral is saturated by random fields. The relevance of an extremal configuration depends on its entropy factor (the quantum fluctations around the extremum effects the contribution of the saddle point). In the configuration series of a Monte Carlo simulation the classical contributions and the quantum fluctations cannot be separated. So we do not intend to identify the gas of quasiparticles found in the Monte Carlo simulation with instantons. In this section we demonstrate only that if the lattice spacing is smaller than the characteristic length of an instanton configuration then the topological charge of the lattice field equals the continuum one.

First we briefly summarise the definition of the topological charge in the continuum model. The continuum $CP^{n-1}$ theory contains the n component complex vector field z (x) of unit length and the gauge field $A_\mu(x)$. The action is

$$S = \frac{n}{g^2} \int_x {}^2 d^2x \, \overline{D_\mu z} \, D_\mu z,$$

where the U(1) covariant derivative is $D_\mu = \partial_\mu - iA_\mu$. One can eliminate the gauge field using the equation of motion

$$A_\mu(x) = i/2 \, [z(x) \, \partial_\mu \bar{z} \, (x) - \bar{z}(x) \, \partial_\mu z(x)], \tag{7}$$

(this procedure is exact for fields appearing quadratically in the action). In this case the Lagrangian is formally the same as before only $A_\mu$ is no longer an independent quantity but it is given by (7). The action is quartic in the field z in this case.

In the continuum theory one must introduce ultraviolet and infrared cutoff in order to make the partition function and the Green functions finite. Let us introduce finite space-time volume as infrared cutoff and confine ourselves to configurations which have finite action in the large volume limit. In this case the fields have to approach a pure gauge configuration at space-time infinity

$$z\,(x) = z_o e^{i\theta(x)} + o\,(|x|^{-1}),$$

$$A_\mu(x) = \partial_\mu \, \theta(x) + o\,(|x|^{-1}).$$

Since the field z is a single-valued function of the space-time coordinates the line integral of the gauge field A along a large circle must be $2\pi \, Q_c$ with $Q_c$ an integer called the topological charge:

$$\int dx^\mu A_\mu = \int d^2x \text{ curl } A = 2\pi \, Q_c + o\,(1)$$

Here the quantity $q = \text{curl} A$ is the topological charge density. In this way the subset of configurations having finite action is divided into subclasses which are characterised by the integer $Q_c$. It is easy to see that the solution of the self duality criterion and the equation of motion in Euclidean space-time give the absolute minima of the action in a given $Q_c$ subclass [5]. One can find the most general configuration of this kind and it turns out that they are the pure multi instanton or antiinstanton configurations.

Returning to the lattice we remark that assuming $z_{\vec{n}+\hat{\mu}} = \exp(-i\phi_\mu(\vec{n}))\, z_{\vec{\mu}} + \varepsilon$ with $\phi$ and $\varepsilon$ small, the expression (3) gives the topological gauge field (7) in the classical limit. In this way $\phi$ gives the naive generalisation of the topological gauge field on the lattice.

The $Q = 1$ instanton solution of the continuum model is

$$z^i(x) = \frac{\lambda\, u^i + (x^o + i x^1)\, v^i}{(\lambda^2 + x^{o2} + x^{12})^{1/2}}, \qquad (8)$$

where $u^i$ and $v^i$ are orthogonal complex n-vectors of unit length ($\bar{v}v = \bar{u}u = 1, \bar{u}v = 0$), and $\lambda$ is the scale parameter of the solution. The line integral of the gauge field A (which is given by (7)) along a circle with radius R is

$$Q_c' = 2\pi\, ((\lambda/R)^2 + 1)^{-1}$$

This shows the winding $2\pi$ in the phase at $R > \lambda$ where the configuration approaches the pure gauge one.

Let us introduce a fine mesh on this configuration (we want to represent instantons with a length scale greater than the lattice spacing). We find no quasiparticles on this lattice since the plaquette is too small to feel the winding of the phase. But the situation changes when we implement periodic boundary conditions. The continuum solution (8) restricted to finite space-time volume shows periodicity models a gauge transformation (in fact instead of periodicity it has a more restricted behaviour at infinity, the infinite circle is just one point up to a gauge transformation). Apply a gauge transformation along the boundary of the finite lattice in order to arrive at $z(\vec{n}) = z_o$ along the perimeter (see Fig. 2). Starting at a point of the boundary and going along the perimeter the change in the phase of the field z accumulates during the gauge transformation. It is easy to see that whenever it takes on the value $\pm\pi$ one creates a pair of quasiparticles. One quasi-particle of this pair with the appropriate sign of P belongs to the lattice. Arriving to the last point before closing the line of the gauge tranformation we create Q quasiparticle on the finite lattice in the Q-instanton case. This procedure shows that the boundary conditions allow the quasiparticles which are local objects, to feel the large scale structure of the field configuration.

One can be more quantitative in the case of $CP^{2-1}$. Choosing a gauge where $z^2$ is positive one can parametrise the field z as

$$z^1 = \cos(\theta/2)\, e^{i\phi},\; z^2 = \sin(\theta/2),\; 0 \leqslant \theta,\ \phi < 2\pi. \tag{9}$$

The usefulness of this parametrisation is that for any solution of the self duality equation and the equation of motion the combination

$$W = \mathrm{ctg}\,(\theta/2)\, e^{i\phi}, \tag{10}$$

is a holomorfic function of the complexified space-time coordinates $\xi = x^0 + ix^1$ or $\bar{\xi}$ [5]. For the Q-instanton solution one has

$$W = \prod_{i=1}^{Q} (\xi - a_i)/(\xi - b_i)$$

Introducing a lattice of fine resolution on this configuration ($a<\min(|a_k-a_l|$, $|b_k - b_l|$, $|a_j - b_l|)$ one can verify that the only plaquettes with $P_n = 1$ are those which contain a pole $b_k$. For antiinstantons W is a holomorfic function of $\xi$ so we get $P_{\vec{n}} = -1$ at each pole again.

This comparison shows that $Q = \sum P_{\vec{n}}$ gives the total topological charge correctly and in a specific gauge the instanton gas corresponds to quasiparticle gas. This connection does not hold in the opposite direction, we do not know the relation between the dense gas of quasiparticles and the saddle points of the action.

To close this Section we mention what a single meron looks like on the lattice. The meron configuration is characterised by the property that the action diverges logarithmically with the space-time volume: $S_m \sim c \ln V$ [6]. Usually we have a factor $1/g^2$ in front of the action so the meron contribution to the partition function is proportional to $\exp[(1-c/g^2)\ln V]$. Consequently for $g^2 > c$ these configurations may be relevant in the path integral.

The meron configuration of the $CP^{n-1}$ model is known [7]. Using the parametrisation (9).(10) it is given by

$$W = \xi/|\xi|.$$

Introducing a mesh on such a configuration one gets curl $\phi \sim \pi$ for the plaquette containing the point $\xi = 0$. This configuration is irregular so we expect that it is suppressed in the continuum limit.

## 4. RESULTS OF THE MONTE CARLO CALCULATION

In our Monte Carlo calculation we use Metropolis procedure where an update of the field z at the point $\vec{n}$ means that we change two randomly selected real components of $z^i_{\vec{\mu}}$ keeping the sum of squares unchanged. This extremly local updating algorithm allows us to keep the acceptance ratio higher than 0.2 for $n \leq 6$.

Fig. 3 shows the distribution of curl $\phi$ on 20 × 20 lattice at different $g^2$ for $CP^{3-1}$. It supports the picture described in Sect.3, namely the dynamics enhances configurations having curl $\phi$ values in the vicinity of an integer multiple of $2\pi$. One can see how the irregular configurations and among them those which contain meron type plaquettes die out in the small $g^2$ limit.

Fig. 4 displays the quasiparticle structure of the z configuration at different values of the coupling constant. Observe that for large $g^2$ the irregular configurations become dominant since their entropy is large. Fig. 5 shows the distribution of the topological charge at $1/g^2 = 2.2$. The equation of motion for the field A

$$\langle \bar{z}_{\vec{n}} U_\mu(\vec{n}) z_{\vec{n}+\hat{\mu}} \rangle = ( z_{\vec{n}+\hat{\mu}} \bar{U}_\mu(\vec{n}) \bar{z}_{\vec{n}} ),$$

shows that complex conjugation (CP inversion) is realised by the dynamics as a symmetry operation. This symmetry guarantees that the center of the distribution is at Q = 0 and $\langle Q \rangle = 0$. The width of the distribution is found to be proportional to the volume in the range of 15 × 15 - 50 × 50 lattices.

Fig. 6 shows the topological susceptibility as the function of the coupling constant. For $1/g^2 < 2.0$ we observe a gradual onset of polynomial behaviour in $1/g^2$ (in this strong coupling region the topological susceptibility becomes the same for periodic and for free boundary conditions). At $1/g^2 > 2.6$ we get a deriation from the g dependence dictated by (2). In the range of $2.1 < 1/g^2 < 2.6$ the correlation length changes by a factor of four according to (2) so one expects that for $1/g^2 > 2.6$ the $20 \times 20$ lattice becomes too small to display nonperturbative effects.

The error bars on the topological susceptibility are meant to indicate only the order of magnitude of the error because these quantities are flucutating widely during the Monte Carlo process. In fact, since the contribution to Q is integer we have to wait at least $1/\langle Q^2 \rangle$ iteration to get one event. Thus one needs extremely long runs to determine these averages in the small $g^2$ region. The fit of the exponential function prescribed by the large n expansion (see Sect. 5) gives the result

$$\chi_t = (2437 \pm 300)\ \Lambda_L^{\ 2}.$$

The same calculation using free boundary conditions lead to a highly divergent topological susceptibilty in the continuum limit. This is expected since in that case $\langle Q^2 \rangle$ is not saved against perturbative contributions. The dispersion of the topological charge defined by the field U was in slight disagreement with the expected continuum behaviour. To see whether this quantity remains finite in the continuum limit one needs a larger lattice.

We carried out the calculation of the topological susceptibility in the case of the $CP^{2-1}$ model [8]. There the slope of $\langle Q^2 \rangle$ is $c \sim 3.89$ in surprisingly good agreement with [9] (whose slope in our units is $c \sim 3.75$). It is worthwhile to note here that in ref. (9) it is claimed that the topological susceptibility diverges for $CP^{n-1}$ with $n \leqslant 3$. The reason we get the correct continuum limit $n = 3$ is that we have a different action.

## 5. CHARACTERISTIC SCALES OF THE $CP^{n-1}$ MODEL

In order to compare characteristic scales of two different regularisation schemes one has to calculate a physical quantity in both schemes and equate them. We used the dynamically generated mass gap of the auxiliary noncompact field [5] as the physical quantity. The existence of this mass gap renders the integrals infrared finite similarly to of the background field method. First we review the calculation of the mass gap in the continuum case and then we present a similar calculation on the lattice.

Using the Fourier representation of the Dirac delta in the path integral one arrives to a quadratic action in the field z [5]:

$$S = \int d^2x\, \partial_\mu \bar{z}\, \partial_\mu z + i \int d^2x \frac{\alpha}{\sqrt{n}} (|z|^2 - \frac{n}{2g^2})$$

$$- \int d^2x\, [\frac{1}{n} A_\mu A_\mu - \frac{i}{\sqrt{n}} A_\mu(\bar{z}\, \partial_\mu z - z\partial_\mu \bar{z})] - m^2 \int d^2x\, |z|^2$$

Here fields z and A are rescaled by the factors $\sqrt{n}/2g^2$ and $n/2g^2$, respectively, in order to obtain finite propagators in the large n limit. The term containing m introduces an overall multiplicative factor to Z. It can be interpreted as a shift of the domain of the $\alpha$ integral: $D[\alpha] \to D[\alpha + im^2\sqrt{n}]$. This arbitrariness was introduced in order to find the saddle point of Z.

Carrying out the Gaussian integral over the field z one gets

$$Z = \int D[\alpha]\, D[A] \exp\{-n \operatorname{Tr} \ln \Delta - \frac{i\sqrt{n}}{2g^2} \int d^2x\, \alpha(x)\},$$

where

$$\Delta = - \bar{D}_\mu D_\mu + m^2 - \frac{i}{\sqrt{n}} \alpha.$$

The effective action of the fields $\alpha$ and A can be expanded in $1/\sqrt{n}$:

$$-n \text{ Tr } \ln \Delta - \frac{i\sqrt{n}}{2g^2} \int d^2x\, \alpha(x) = \sum_k n^{1-\frac{k}{2}} S^{(k)} + \text{const.}.$$

The first term is

$$S^{(1)} = \frac{i}{2g^2} \int d^2x\, \alpha(x) - i \int d^2x\, \alpha\,(x) \cdot (- \quad + m^2)^{-1}_{x,x}.$$

$S^{(2)}$, which is quadratic in the fields, gives the propagators in this expansion. The vanishing of $S^{(1)}$ gives the condition of the saddle point:

$$\frac{1}{2g^2} = \int \frac{d^2g}{(2\pi)^2} (m^2 + q^2)^{-1}. \tag{11}$$

The integral appearing in this equation is ultraviolet divergent; applying Pauli-Villard regularisation one gets

$$\frac{1}{2g^2} = \frac{1}{4\pi} \ln \frac{\Lambda^2}{m^2}$$

We define $\Lambda_{PV}$ as $\Lambda_{PV} = \Lambda\, e^{-\pi/g^2}$.

Now we turn to the calculation of the mass gap with lattice regularisation. One can follows essentially the same procedure as before. The only difference is that we parametrise the U(1) gauge field as

$$U_\mu(\vec{n}) = e^{i A_\mu(\vec{n})}$$

and we expand the exponential function. The integration over the field z can be done by introducing an auxiliary field $\alpha_{\vec{n}}$ just as before. The saddle point condition is now

$$\frac{1}{2g^2} = \frac{1}{4\pi}\frac{1}{V}\sum_{\vec{k}} (m^2 + 4 - 2\cos k_1 - 2\cos k_2)$$

which is the same as equation (11) but in lattice regularisation. The asymptotic behaviour of the sum for small m is [12]

$$\frac{\Lambda}{2g^2} = \frac{1}{4\pi}\ln (am)^{-2} + \frac{1}{4\pi}\ln c$$

where c = 32. For a finite lattice c depends on the coupling constant. This function is plotted on Fig. 7 for different lattice sizes. The surprising feature of this function is that it takes its asymptotic value only for relatively large (e.g. larger than 50 × 50) lattice. This means that one has to take a large lattice in order to see the correct continuum behaviour in terms of the unconstrained auxiliary field z. Fortunately the constrained field z appearing in the original action (1) is different from the unconstrained auxiliary one so we have chance to reach the continuum limit on a 20 × 20 lattice in our calculation.

Given a value of c the ratio $\Lambda_L/\Lambda_{PV}$ is determined by the requirement

$$m = \Lambda\, e^{-\frac{\pi}{g_c^2}} = \sqrt{c}\, a^{-1} e^{-\frac{\pi^2}{g_L^2}} .$$

Consequently we have

$$\Lambda_L/\Lambda_{PV} = 1/\sqrt{c}. \tag{12}$$

The topological susceptibility has been calculated in the leading order of 1/n [5]

$$\chi_t = \frac{3}{n\pi} \cdot \Lambda^2{}_{PV}.$$

Comparing this relation with the result of the Monte Carlo calulation we arrive at a huge discrepancy if c takes its asymptotic value. This discrepancy originates from finite volume effects, and it excludes the comparison of the scales of the 20 × 20 lattice and the 1/n expansion. Nevertheless we can compare at least the order of magnitude of the scales restricting ourselves to a very small region of the coupling constant $g^2$. Approximating c by a constant we have $c \sim 10^3$ which after substitution in (12) gives the right order of magnitude for the ratio.

6. VORTEX LIKE GAUGE TRANSFORMATIONS IN THE CONTINUUM LIMIT

It is a well known fact that taking the continuum theory on a lattice one occasionally gets short wavelength low energy modes which are not present in the original continuum version of the theory. Such examples are the localized topological excitations in lattice U(1) gauge theory [13] and the extra minima in the dispersion relation of free fermions on the lattice. Although these excitations have different origins both of them survive the continuum limit. In fact the density of the localized topological excitations is finite near the critical point of the lattice U(1) gauge theory and the extra degeneracy factors accompanying the fermion propagators are present in the continuum limit.

The lattice $CP^{n-1}$ model posses such a lattice mode. In fact the vortex like gauge transformations connect those configurations whose action becomes different in the continuum limit. The reason is that the variables of the continuum theory (e.g. the topological vector field $A_\mu = \frac{i}{2} \bar{z} \overleftrightarrow{\partial} z$) have no compact nature. In this Section we show that the effect of this lattice mode is that the usual weak coupling and the $1/n$ expansion is singular in the gauge variant sector of the theory even in the case of complete gauge fixing. This phenomenon illustrates an important role of the gauge invariance, namely it selects quantities which have smooth $g^2$ or $1/n$ dependence.

We indicated in Sect. 2 that the action of the lattice $CP^{n-1}$ model enhances configurations having curl $\phi \sim 2\pi k$. We argue first that this weighting factor is periodic in curl $\phi$ with period length $2\pi$. In fact one can integrate out all variables but $\phi$ in (1). The remaining effective action depends on the product of exp $i\phi$ along closed loops since any other combination is gauge dependent. (It is worthwhile to note that the effective action of the variable A is calculable in the large n and large g limit. Expanding the determinant of $\Delta$ in the U field instead of A one gets result similar to the hopping parameter expansion [10].) The product of exp $i\phi$ along a closed loop is the product of exp(icurl $\phi$) inside the loop. Consequently the effective action is periodic in curl $\phi$. The distribution of curl $\phi$ at a plaquette is the product of the integration measure and the weight described by the effective action. Since the latter is periodic in curl $\phi$ the effect of the coupling constant $g^2$ is to amplify or to smear out the enhancement at curl $\phi \sim 2\pi k$. The relative area of the peaks in the distribution function is determined by the integration measure and not by the effective action. (This fact was confirmed by the Monte Carlo simulation. The ratio of space-time containing quasiparticles is .33 in the extreme small $g^2$ region.) In the presence of a $g^2$ independent gauge fixing condition the above argument remains valid because the gauge fixing term can be inserted into the integration measure. In this way we conclude that the density of quasiparticles remains finite in the small $g^2$ limit on a finite lattice even in the case when the gauge is completely fixed.

Now we turn to study the effect of a vortex like gauge transformation in the framework of the small $g^2$ or large n expansion. In the case of the weak coupling expansion the gauge is fixed by requiring that $z^1$ is real and positive. In this gauge the field z can be written as

$$z^1 = \sqrt{1 - \bar{u}\,\bar{u}}\ , \quad z^{i+1} = u^i$$

where $u^i$ is a complex vector field of n-1 components. Rescaling the fields $u \to \sqrt{n/2g^2}\, u$, $A \to \sqrt{n/2g^2}\, A$ the action reads as

$$S = - \sum A_\mu^2(\vec{n}) - \sum (2\, \bar{u}_{\vec{n}}\, u_{\vec{n}} - \bar{u}_{\vec{n}} u_{\vec{n}+\hat{\mu}} - \bar{u}_{\vec{n}+\hat{\mu}}\, u_{\vec{n}}) + 0(g^2)$$

In the small $g^2$ limit the phase variable

$$\phi = \frac{1}{i} \ln \Big\{ \Big[ \Big(1 - \frac{2g^2}{n}\ \bar{u}_{\vec{n}}\, u_{\vec{n}}\Big)^{1/2} \Big(1 - \frac{2g^2}{n}\, \bar{u}_{\vec{n}+\hat{\mu}}\, u_{\vec{n}+\mu}\Big)^{1/2} + \frac{2g^2}{n}\, \bar{u}_{\vec{n}}\, u_{\vec{n}+\mu} \Big] \Big/$$

$$\Big| \Big[ \Big(1 - \frac{2g^2}{n}\, \bar{u}_{\vec{n}}\, u_{\vec{n}}\Big)^{1/2} \Big(1 - \frac{2g^2}{n}\, \bar{u}_{\vec{n}+\mu}\, u_{\vec{n}+\hat{\mu}}\Big)^{1/2} + \frac{2g^2}{n}\, \bar{u}_{\vec{n}}\, u_{\vec{u}+\hat{\mu}} \Big] \Big| \Big\}$$

becomes smooth rendering a vanishing quasiparticle density.

In the large n expansion the original z field is substituted by the unconstrained one so the field $U_\mu$ remains only to see the quasiparticles. In the large n expansion outlined in Sect. 4 we get a term (curl $A_\mu)^2$ in the next to the leading order of the effective action. This is the noncompact approximation of the function 1 - cos (curl $A_\mu$) and this action suppresses quasiparticles again. The situation can be summarised as follows: The expansion of the square root in the weak coupling expansion or the exponential function exp(iA) in the 1/n scheme destroys the compact nature of the functions. As a result any effect which originates from this compactness is missed by these expansions. In addition these expansions break the gauge invariance of the theory.

The vortex like gauge transformations produce an other difference between the continuum and the lattice gauge models. In the continuum theory the Wilson loop of a fractional charge is gauge invariant. But on the lattice the vortex like gauge transformation changes its value without changing the action. Consequently this type of Wilson loop ceases to be physical quantity. This fact is relavant in the building of realistic gauge theories on a lattice since one has to use integer charged particles only.

7. SUMMARY

We have demonstrated that configurations which dominate the path integral (1) in the continuum limit show certain quasiparticle like structures. We introduced a quantity which reduces to the topological charge density of the continuum $CP^{n-1}$ model in the classical continuum limit. It was showed that the space-time sum of this density divided by the volume of the gauge group, Q is an integer on any periodic lattice field configuration. In addition we pointed out that with a continuum instanton configuration, introducing a mesh in space-time, the resulting lattice field configuration gives the same total topological charge if the lattice spacing is smaller than the characteristical length of the continuum configuration. The topological susceptibility was found to be finite in the continuum limit. We calculated the ratio $\Lambda_L / \Lambda_{PV}$, and found that it has not yet reached the asymptotic value on 20 × 20 lattice. Hence the absolulte scale of the topological susceptibility can not be determined from the present calculation. We pointed out that the existence of vortex like gauge transformations produces a singularity at $g^2 = 0$ in the gauge dependent sector of the theory even in the case of complete gauge fixing. Another effect of these gauge transformations is that there are no fractional charged particles in a compact lattice gauge theory.

I thank Peter Hrasko, Julius Kuti, Martin Luscher for critical remarks and Paolo di Vecchia, Jorn Knoll for helpful discussions. I am indebted to the Rechenzentrum of the Gesellschaft fur Schwerionenforschung for providing me with optimal circumstances in the numerical work.

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Table 1

| $1/g^2$ | $\langle Q^2 \rangle$ |
|---|---|
| 1.6 | 7,62.$10^{0}$ |
| 1.7 | 5,90.$10^{0}$ |
| 1.8 | 4,53.$10^{0}$ |
| 1.9 | 3,69.$10^{0}$ |
| 2.0 | 2,38.$10^{0}$ |
| 2.1 | 1,72.$10^{0}$ |
| 2.2 | 9.30.$10^{-1}$ |
| 2.3 | 5,95.$10^{-1}$ |
| 2.4 | 2,29.$10^{-1}$ |
| 2.5 | 1,17.$10^{-1}$ |
| 2.6 | 7,82.$10^{-2}$ |
| 2.7 | 2,54.$10^{-2}$ |
| 2.8 | 8,64.$10^{-3}$ |
| 2.9 | 7,24.$10^{-3}$ |
| 3.0 | 2,25.$10^{-3}$ |
| 3.1 | 5,65.$10^{-5}$ |

FIGURE CAPTIONS

All data refers to 20 × 20 lattice of the $CP^{3-1}$ model.

Fig. 1. The effect of a vortex like gauge transformation on the $z = z_0$ configuration. The arrays show the phase of the field z:

$$u_{\vec{n}} = -i \ln \left(\sum_i z^i_{\vec{n}} / \left|\sum_i z^i_{\vec{n}}\right|\right)$$

Fig. 2. Introducing a fininte lattice on a Q-instanton configuration and applying gauge transformation along the perimeter in order to arrive periodic configuration we create Q quasiparticles.

Fig. 3. The distribution of curl at $a{:}1/g^2 = 0.6$, $b{:}1/g^2 = 2.0$, $c{:}1/g^2 = 2.6$.

Fig. 4. Snap shot of the value of curl $\phi$ in thermal equillibrium. The definition of the symbols is:

$-8\pi/2 < \text{curl}\ \phi < -7\pi/2$ : =

$-7\pi/2 < \text{curl}\ \phi < -5\pi/2$ : "

$-5\pi/2 < \text{curl}\ \phi < -3\pi/2$ : -

$-3\pi/2 < \text{curl}\ \phi < -\pi/2$ : a

$-\pi/2 < \text{curl}\ \phi < \pi/2$ :

$\pi/2 < \text{curl}\ \phi < 3\pi/2$ : m

$3\pi/2 < \text{curl}\ \phi < 5\pi/2$ : *

$5\pi/2 < \text{curl}\ \phi < 7\pi/2$ : :

The values of the coupling constant are: $a{:}1/g^2 = 2.6$, $b{:}1/g^2 = 2.2$, $c{:}1/g^2 = 1.8$, $d{:}1/g^2 = 0.2$.

Fig. 5. The distribution of Q at $1/g^2 = 2.2$. We used 800 iterations to calculate this graph. The dashed line is to guide the eye.

Fig. 6. The topological susceptibility as the function of the coupling constant. The datas are the averages of three thousand iterations after at least six thousand sweeps at every $g^2$ value. For later application the result is collected in Table 1.

Fig. 7. The proportionality constant c appearing in the expression of the mass gap as the function of the coupling constant. The asymptotic value is $c = 32$.

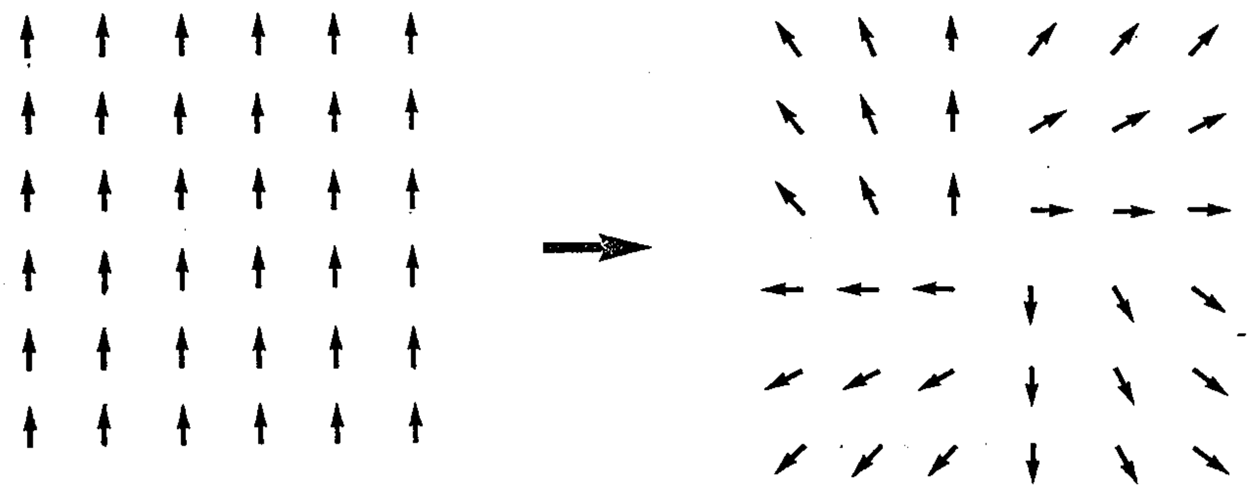


Fig. 1

GSI-C2-82-0004-4

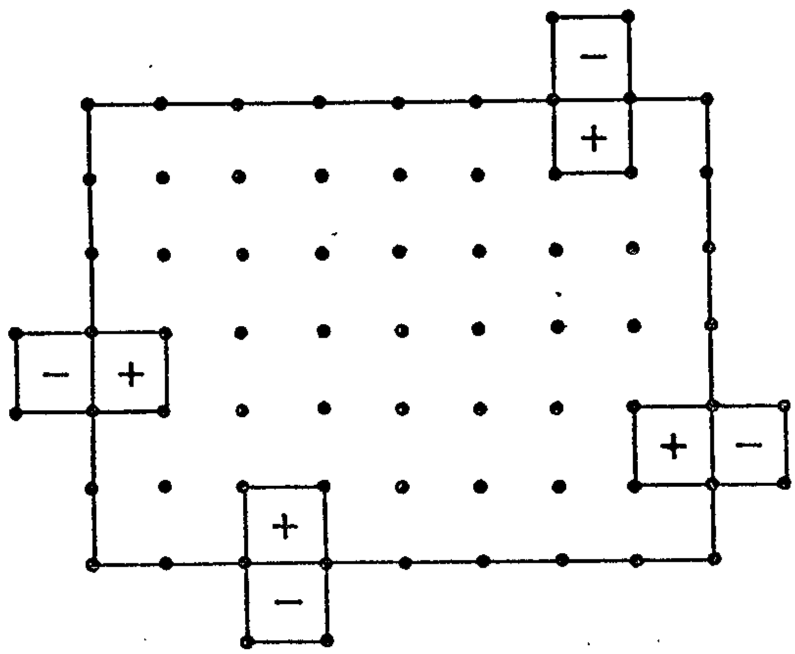



Fig. 2

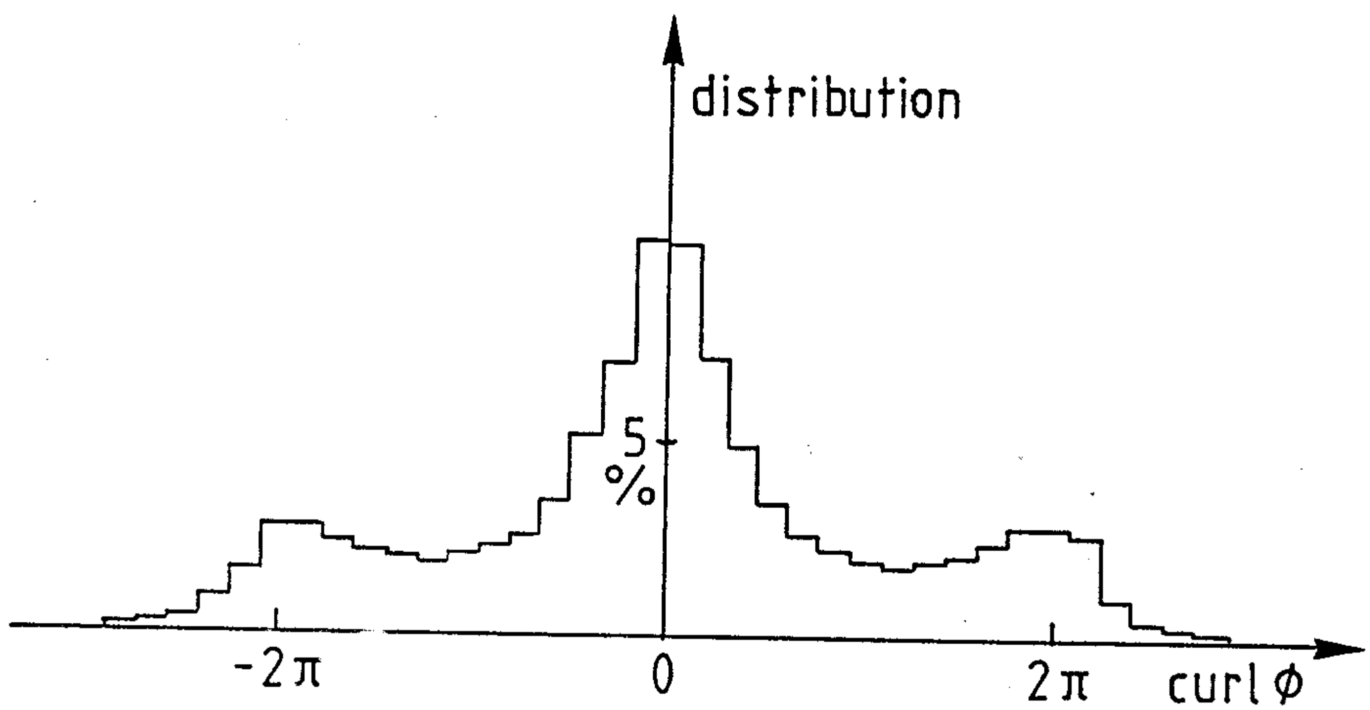
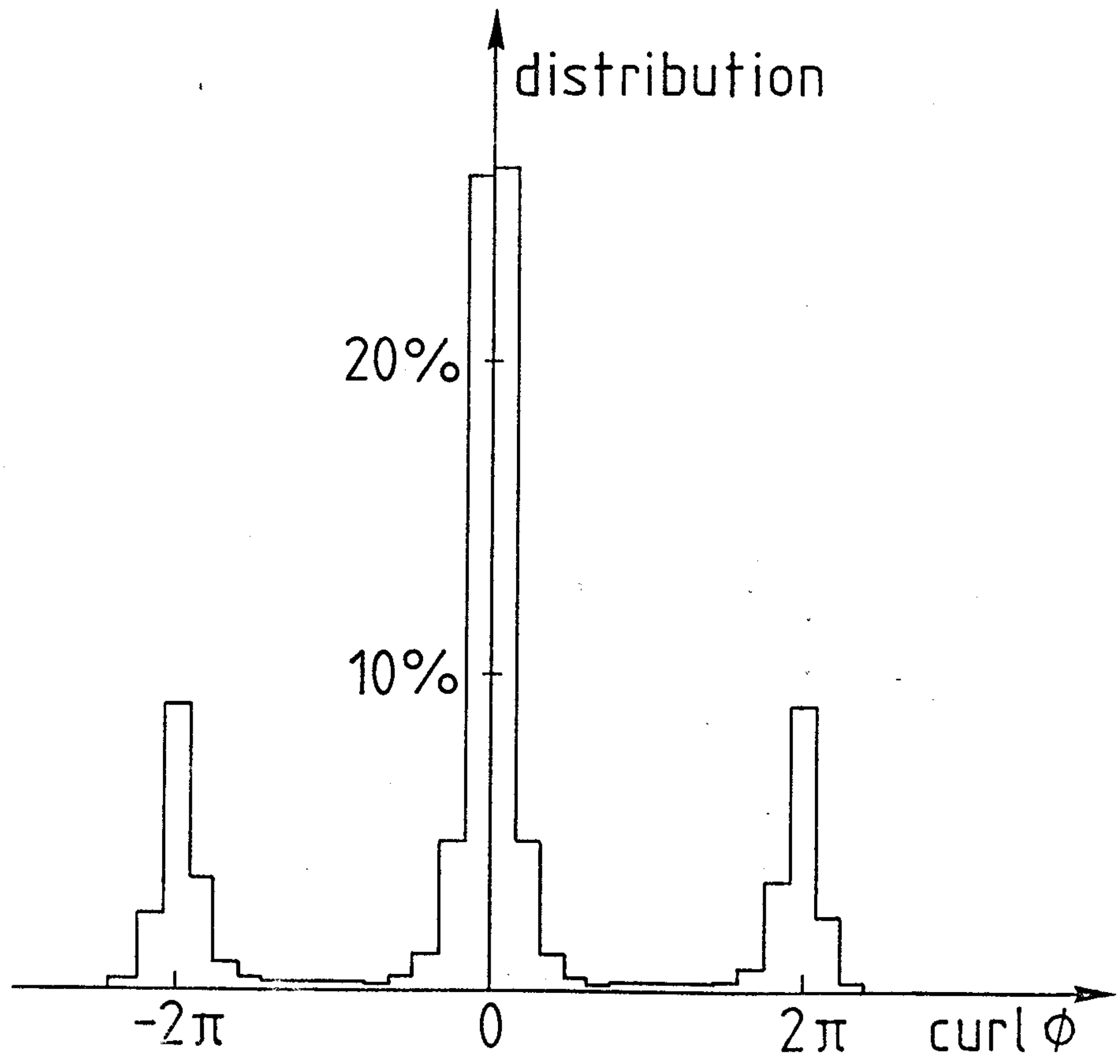



Fig. 3a

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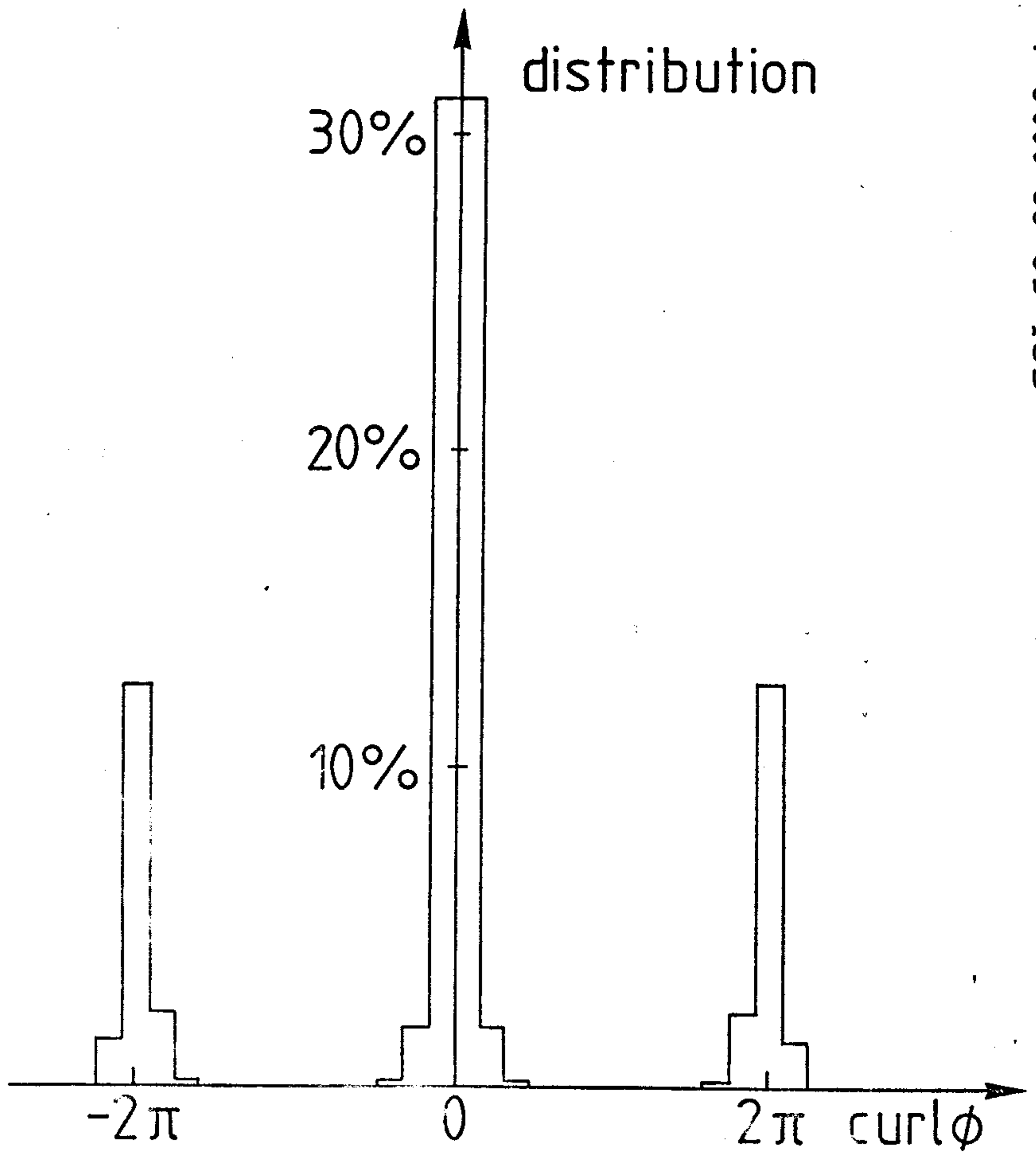



GSI-C2-82-0008-4

Fig. 3c

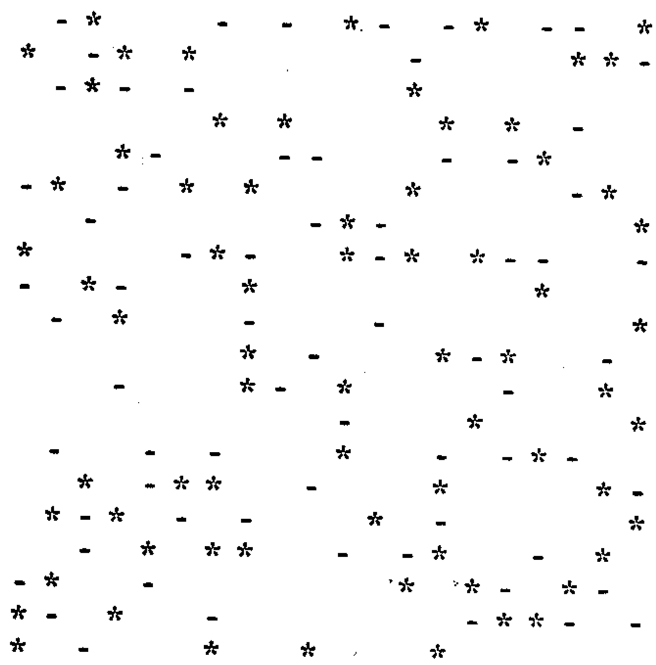


Fig. 4a

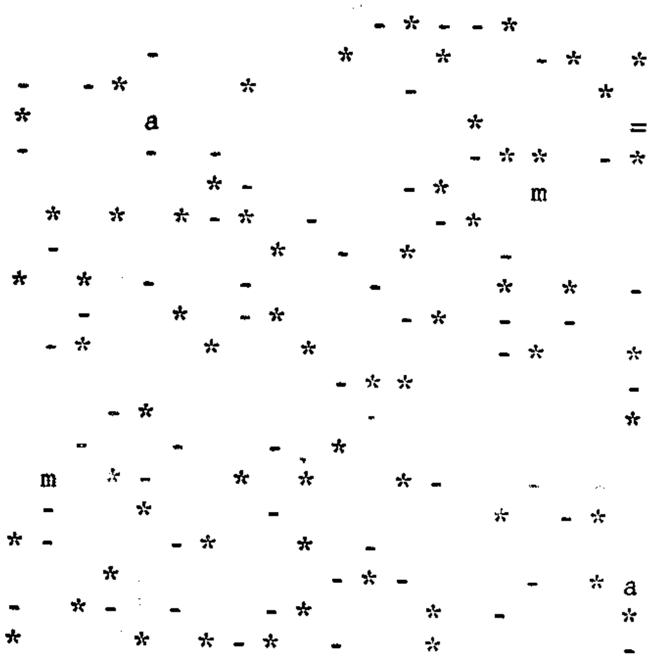



Fig. 4b

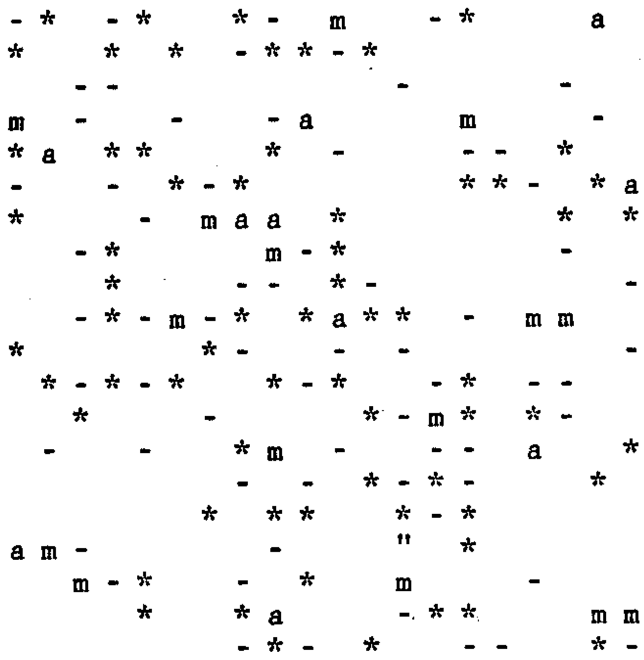


Fig. 4c

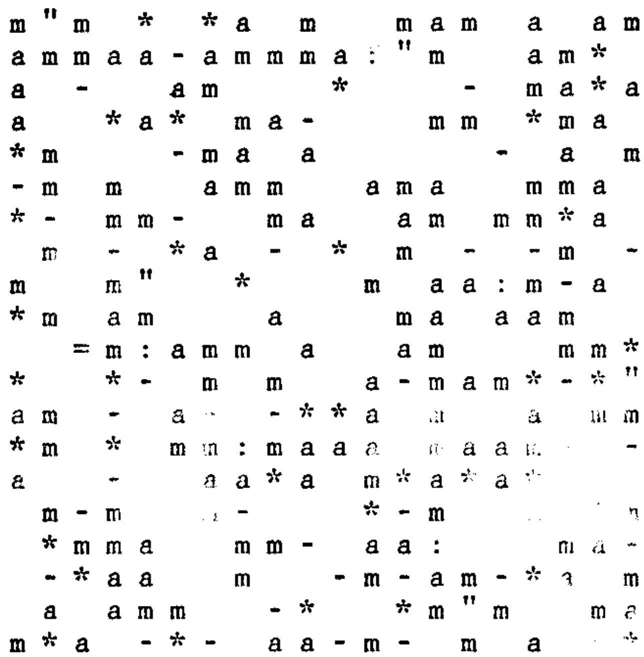
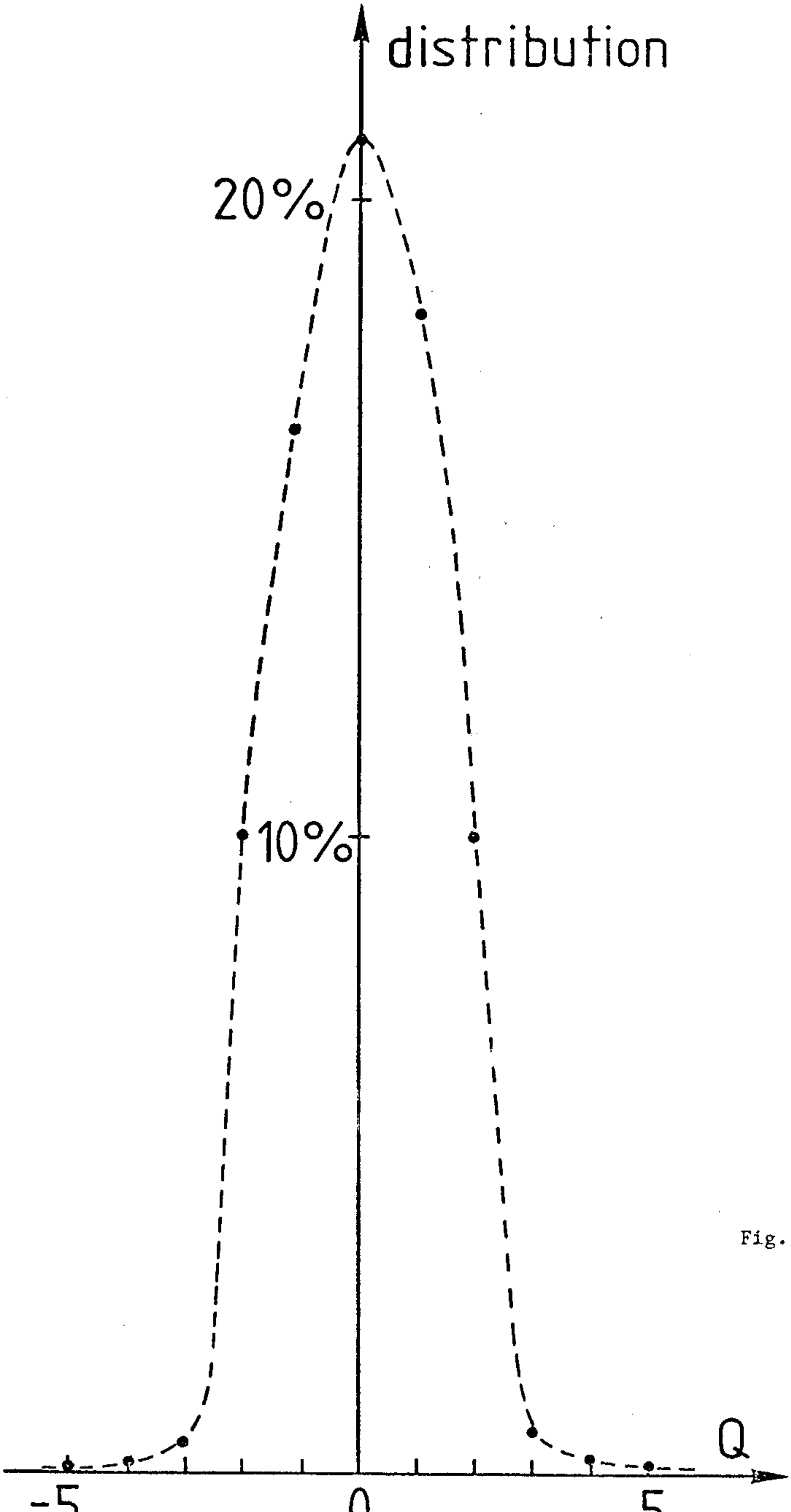
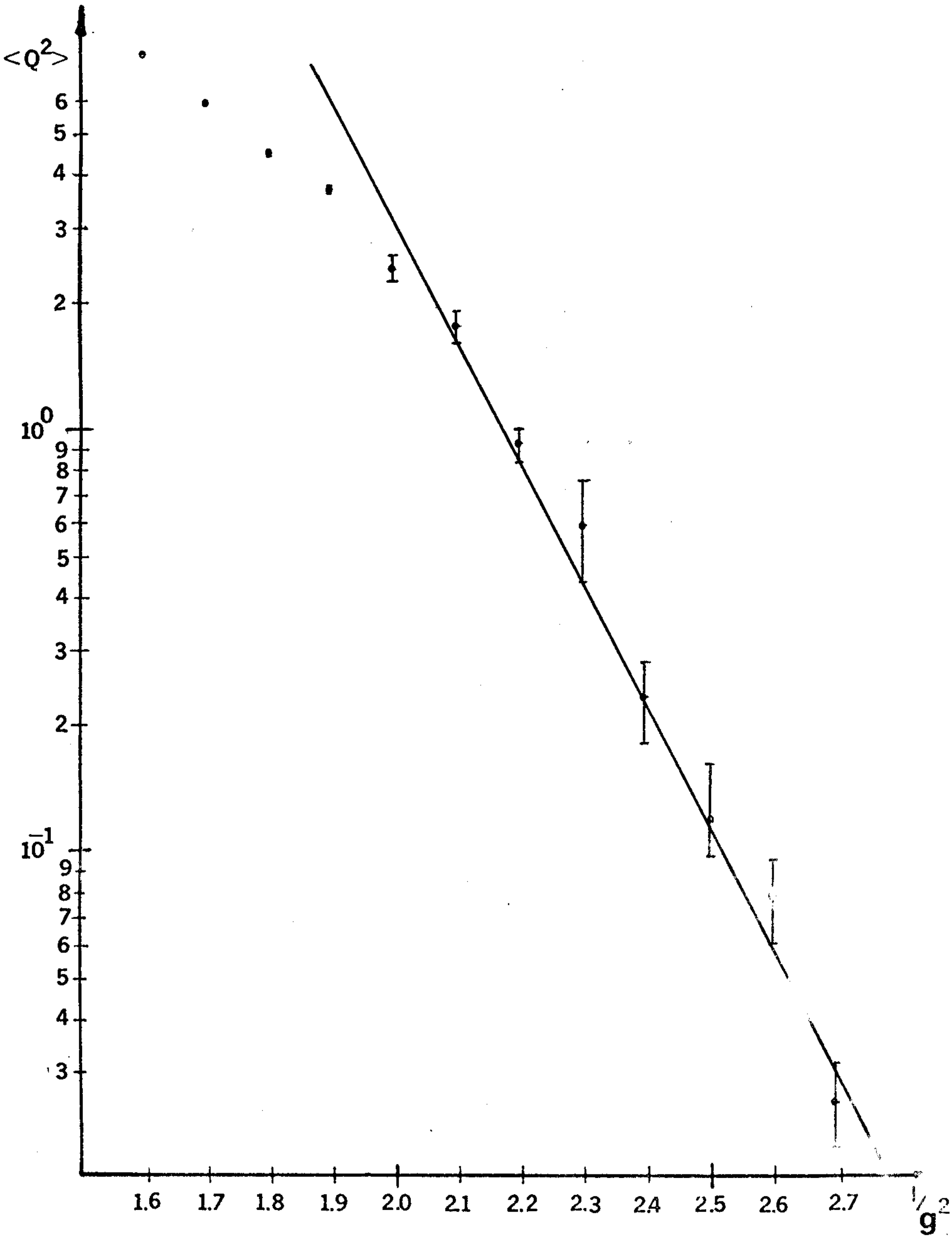


Fig. 4d



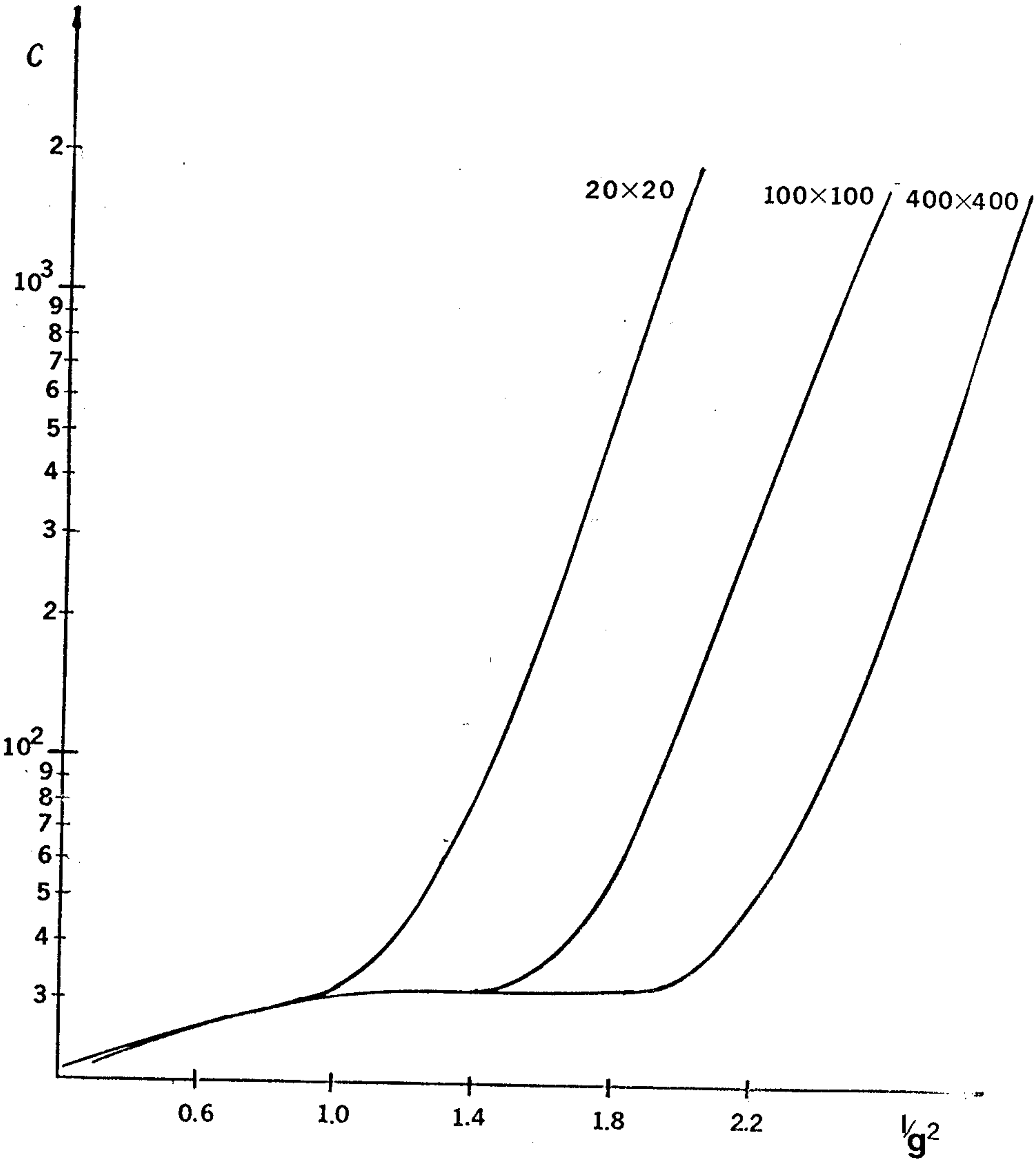

GSI-C2-82-0009-4

Fig. 5



<Q2>
6
5
4
3
2
10^0
9
8
7
6
5
4
3
2
10^-1
9
8
7
6
5
4
3
1.6
1.7
1.8
1.9
2.0
2.1
2.2
2.3
2.4
2.5
2.6
2.7
1/g2


Fig. 6



C
2
$10^3$
9
8
7
6
5
4
3
2
$10^2$
9
8
7
6
5
4
3
20×20
100×100
400×400
0.6
1.0
1.4
1.8
2.2
$1/g^2$


Fig. 7

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